

Approximating Rectangles by Juntas and Weakly-Exponential Lower Bounds for LP Relaxations of CSPs

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Abstract

We show that for constraint satisfaction problems (CSPs), sub-exponential size linear programming relaxations are as powerful as $n^{\Omega(1)}$ -rounds of the Sherali-Adams linear programming hierarchy. As a corollary, we obtain sub-exponential size lower bounds for linear programming relaxations that beat random guessing for many CSPs such as MAX-CUT and MAX-3SAT. This is a nearly-exponential improvement over previous results; previously, it was only known that linear programs of size $n^{o(\log n)}$ cannot beat random guessing for any CSP [CLRS13].

Our bounds are obtained by exploiting and extending the recent progress in communication complexity for "lifting" query lower bounds to communication problems. The main ingredient in our results is a new structural result on "high-entropy rectangles" that may of independent interest in communication complexity.

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1 Introduction

Translating a combinatorial problem over a discrete domain to a problem in continuous space has been an important concept in computer science over the last few decades; in this vein, linear programming relaxation is one of the most used techniques for algorithm design. In this work we prove limitations on the power of linear programs (LPs) as applied to *constraint satisfaction problems* (CSPs).

Constraint satisfaction problems such as MAX-3SAT or MAX-3XOR or MAX-CUT are some of the most well-studied problems in approximation algorithms as well as combinatorial optimization. Here we show unconditional lower bounds for approximately solving CSPs by LPs. Informally, we show that for many CSPs such as MAX-3SAT, MAX-3XOR, or MAX-CUT, no LP of size $2^{n^{\Omega(1)}}$ can beat the *trivial* approximation factor (7/8 for MAX-3SAT, 1/2 for MAX-3XOR, 1/2 for MAX-CUT); we also show similar results for vertex-cover. Previously, such lower bounds only applied to LPs of size at most $n^{\Omega((\log n)/(\log \log n))}$ [CLRS13].

The core of our result above is a new structural result about rectangles that has various applications in communication complexity in the context of *lifting query* lower bounds to communication lower bounds.

1.1 CSPs, Linear programming relaxations, Sherali-Adams hierarchy

A MAX-CSP (henceforth referred to only as CSP) is defined by a *predicate* $P : \{-1, 1\}^k \rightarrow \{0, 1\}$. An instance of the CSP, \mathcal{I} , is defined by a collection of k -tuples of literals C_1, C_2, \dots, C_m on n Boolean variables (x_1, x_2, \dots, x_n) ¹. The algorithmic problem is to find an assignment to the variables $x = (x_1, \dots, x_n)$ so as to maximize the number of satisfied constraints:

$$\text{opt}(\mathcal{I}) = \max_{x \in \{-1, 1\}^n} \sum_{i=1}^m P(C_i(x)) \equiv \max_{x \in \{-1, 1\}^n} \mathcal{I}(x), \quad (1.1)$$

where we define $\mathcal{I}(x) = \sum_{i=1}^m P(C_i(x))$.

For example, MAX-CUT corresponds to the case where the predicate $P : \{-1, 1\}^2 \rightarrow \{0, 1\}$ is defined by $P(a, b) = (1 - ab)/2$ with instances corresponding to graphs.

Here we consider a broad-class of linear programming relaxations for CSPs obtained by *linearizing* the objective function $\mathcal{I}(x)$. Formally, given a predicate P , and an integer D , we want:

Definition 1.1 (Linearization of a CSP). 1. A vector $v_x \in \mathbb{R}^D$ for every $x \in \{-1, 1\}^n$.

2. A vector $w_{\mathcal{I}} \in \mathbb{R}^D$ for every instance \mathcal{I} of the CSP.

3. For every assignment x and every instance \mathcal{I} , $\mathcal{I}(x) = \langle w_{\mathcal{I}}, v_x \rangle$.

Given a linearization as above, we can define a relaxation of the CSP as follows. For a polytope $\mathcal{P} \subseteq \mathbb{R}^D$ with $\{v_x : x \in \{-1, 1\}^n\} \subseteq \mathcal{P}$, we look at the linear program

$$\text{opt}_{\mathcal{P}}(\mathcal{I}) = \max_{y \in \mathcal{P}} \langle w_{\mathcal{I}}, y \rangle.$$

¹Throughout this article, we will use $\{-1, 1\}$ to denote Boolean inputs

Clearly, $\text{opt}(I) \leq \text{opt}_{\mathcal{P}}(I)$. The *complexity* or *size* of the relaxation is defined as the number of facets (or inequalities) needed to describe the polytope \mathcal{P} .

Approximating CSPs by LP relaxations. Consider a CSP defined by a predicate $P : \{-1, 1\}^k \rightarrow \{0, 1\}$. A LP relaxation for the CSP is a sequence of polytopes $\mathcal{P} \equiv \{\mathcal{P}_n : n \geq 1\}$ where for each $n \geq 1$, \mathcal{P}_n is a relaxation for n -variable instances of the CSP as defined above. For a function $s : \mathbb{N} \rightarrow \mathbb{N}$, we say \mathcal{P} has size at most $s(n)$ if each \mathcal{P}_n has size at most $s(n)$.

For $0 < c \leq s \leq 1$, we say \mathcal{P} achieves a (c, s) -approximation for the CSP if for n -variable instances I with $\text{opt}(I) \leq s$, $\text{opt}_{\mathcal{P}_n}(I) \leq c$. Similarly, for $0 \leq \alpha \leq 1$, we say \mathcal{P} achieves a α -approximation if for all $n \geq 1$, $\text{opt}(I) \geq \alpha \cdot \text{opt}_{\mathcal{P}_n}(I)$. In the latter case, we also say \mathcal{P} has *integrality-gap* at most $(1/\alpha)$.

The above framework introduced in the work of [CLRS13] generalizes the *extended formulation* framework of Yannakakis [Yan88] and its adaptation to approximation algorithms as formulated in [BFPS15]. Furthermore, LPs arising out of the Lovasz-Schriever (LS) [LS91] or the Sherali-Adams [SA90] hierarchies are captured within this framework.

We prove that despite their apparent generality, when it comes to CSPs, general linear programs as above, and hence all extended formulations, are only as powerful as those obtained from the Sherali-Adams hierarchy:

Theorem 1.2. *There exist constants $0 < h < H$ such that the following holds. Consider a function $f : \mathbb{N} \rightarrow \mathbb{N}$. Suppose that the $f(n)$ -round Sherali-Adams relaxation for a CSP cannot achieve a (c, s) -approximation on instances on n variables. Then, no LP relaxation of size at most $n^{hf(n)}$ can achieve a (c, s) -approximation for the CSP on n^H variables.*

Charikar, Makarychev, and Makarychev [CMM09] showed that for all $\varepsilon > 0$, there is a constant $\gamma(\varepsilon)$ such that $n^{\gamma(\varepsilon)}$ -round Sherali-Adams relaxation for MAX-CUT has integrality gap at least $2 - \varepsilon$. Similarly, it follows from the works of Grigoriev [Gri01] (and from that of Schoenebeck [Sch08a]) that $\Omega_\varepsilon(n)$ -round Sherali-Adams relaxations have integrality gap at least $2 - \varepsilon$, $8/7 - \varepsilon$ for MAX-3XOR and MAX-3SAT respectively. As a corollary, we get the following lower bounds for solving CSPs by linear programming relaxations.

Corollary 1.3. *For some universal constant $H \geq 1$, for every $\varepsilon > 0$, there exist constants $c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon)$ such that the following hold: no LP relaxation of size less than $2^{c_1(\varepsilon)n^{1/H}}$ has integrality gap less than $(8/7 - \varepsilon)$ for MAX-3SAT; no LP relaxation of size less than $2^{c_2(\varepsilon)n^{1/H}}$ has integrality gap less than $(2 - \varepsilon)$ for MAX-3XOR; no LP relaxation of size less than $2^{c_3(\varepsilon)n^{1/H}}$ has integrality gap less than $(2 - \varepsilon)$ for Max-CUT.*

We also get similar bounds more generally for CSPs defined by *pairwise-independent* predicates by combining Theorem 1.2 with known integrality-gaps for such CSPs ([BGMT12]).

The above results for CSPs are established through a more general claim on *non-negative rank* of a class of matrices referred to as *pattern matrices*. We explain this connection and results next.

1.2 Lifting degree lower bounds to rank lower bounds

In the seminal work introducing extended formulations, Yannakakis showed that the extended formulation complexity of an optimization problem is precisely the *non-negative rank* of the associated slack matrix. In [BFPS15], this connection was subsequently extended to approximation by linear

programs. All known lower bounds on the size of extended formulations rely on this connection as do we.

Definition 1.4 (Non-negative Rank). Let M be a non-negative matrix. The non-negative rank of M , denoted by $\text{rank}_+(M)$ is the least positive integer r such that there exist non-negative rank 1 matrices M_1, \dots, M_r such that $M = \sum_{i=1}^r M_i$.

Proving lower bounds on non-negative rank of specific matrices is often non-trivial; a significant breakthrough towards proving such lower bounds was achieved by the work of [FMP⁺15] who showed a connection between communication complexity lower bounds and non-negative rank.

We give a tight characterization of the non-negative rank of a broad-class of matrices—*pattern matrices*—that were studied before in communication complexity [RM99, Raz03, She11].

Definition 1.5 (Pattern Matrix). Fix positive integers n and q . Given functions $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ and $g : [q] \times [q] \rightarrow \{-1, 1\}$, the composed function $f \circ g^{\otimes n} : [q]^n \times [q]^n \rightarrow \mathbb{R}$ is defined as,

$$f \circ g^{\otimes n}(x, y) \stackrel{\text{def}}{=} f(g(x_1, y_1), \dots, g(x_n, y_n)) ,$$

where we have $x_i, y_i \in [q]$ for all $i \in [n]$. The pattern matrix M_f^g is the truth-table of the composed function $f \circ g^{\otimes n}$ expressed as a matrix, i.e., it is a matrix with rows and columns indexed by $[q]^n$ with,

$$M_f^g(x, y) \stackrel{\text{def}}{=} f \circ g^{\otimes n}(x, y) .$$

The function $g : [q] \times [q] \rightarrow \{-1, 1\}$ is referred to as the gadget function. Throughout this work, we will use the Boolean *inner-product function* as the gadget g . Specifically we will set $q = 2^b$ for $b \in \mathbb{N}$, identify $[q]$ with $\{0, 1\}^b$ and define

$$g : \{0, 1\}^b \times \{0, 1\}^b \rightarrow \{-1, 1\} \text{ given by } g(x, y) \stackrel{\text{def}}{=} (-1)^{\bigoplus_{i=1}^b x_i y_i} .$$

With this choice of the gadget function g , we will use M_f^b to denote the pattern matrix M_f^g ; we also drop the superscript b and use M_f to denote M_f^b when b is clear from context.

Our main result characterizes the non-negative rank of pattern matrices M_f by a corresponding measure of f that we define next.

Definition 1.6 (Juntas and Non-negative Degree). A function $h : \{-1, 1\}^n \rightarrow \mathbb{R}$ is a d -junta if it only depends on at most d coordinates. A function $h : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ is a *conical* d -junta if it can be written as a non-negative linear combination of non-negative d juntas.

For any $f : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$, the non-negative degree of f , written as $\deg_+(f)$, is the least positive integer d such that f can be written as a conical d -junta.

We show that for any non-negative function f , the non-negative rank of M_f is essentially characterized by the non-negative degree of f . Indeed, it is easy to check that

$$\text{rank}_+(M_f^b) \leq \binom{n}{\deg_+(f)} \cdot 2^{b \cdot \deg_+(f)} . \tag{1.2}$$

We show a nearly matching lower bound for $\text{rank}_+(M_f^b)$; specifically, we show that if small positive shifts of f have² high non-negative degree, then $\text{rank}_+(M_f)$ is correspondingly large.

Theorem 1.7 ($\text{rank}_+(M_f)$ vs $\deg_+(f)$). *There exist constants $c, C > 0$ such that the following holds. Let $f : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ be such that $\mathbb{E}[f] = 1$. Then,*

$$\text{rank}_+(M_f^b) \geq 2^{c \cdot b \cdot \deg_+(f + \eta)}.$$

for all $\eta \geq 1/n$ and $b \geq C(\log n)$.

Note that by Equation 1.2, for $b > \log n$, $\text{rank}_+(M_f^b) \leq 2^{2b \cdot \deg_+(f)}$. Thus, the above theorem is tight up to constant factors (in the exponent) and working with $\deg_+(f + \eta)$.

1.3 Previous work: Approximate non-negative rank versus non-negative rank

The above result should be compared with similar results in [GLM⁺15, LRS15]. Although they also obtain similar lifting theorems, a crucial difference is that they lower bound the *approximate non-negative* rank of lifted matrices. For a non-negative matrix M , and $\varepsilon > 0$, define the ε -approximate non-negative rank as

$$\text{rank}_+^\varepsilon(M) = \min\{\text{rank}_+(M') : \|M' - M\|_\infty \leq \varepsilon \|M\|_\infty\}.$$

Clearly, $\text{rank}_+^\varepsilon(M) \leq \text{rank}_+(M)$ for all $\varepsilon > 0$. At a high-level, the previous works show lower bounds on $\text{rank}_+^\varepsilon(M_f)$ (in terms of the *approximate non-negative junta* degree of f). Similarly, while [GLM⁺15] show a separation between $\text{rank}_+^\varepsilon, \text{rank}_+^\delta$ for some constants $0 < \varepsilon < \delta < 1$, the resulting matrices have large rank. Such lifting theorems are not enough to obtain our applications to CSPs – Theorem 1.2, Corollary 1.3 – as matrices arising in these applications in fact have small approximate non-negative rank (roughly $n^{O(\log(1/\varepsilon))}$) and small rank. This was one of the main reasons why the previous works only obtained quasi-polynomial size lower bounds.

In fact, before our work, the best separation between $\text{rank}_+^\varepsilon$, rank and rank_+ was only quasi-polynomial. As a corollary of our results, we obtain weakly-exponential separation for an explicit matrix:

Theorem 1.8. *For all $\varepsilon > 0$, there exist constants $0 < c_\varepsilon, C_\varepsilon$ such that the following holds. There exists an explicit non-negative matrix $M \in \mathbb{R}_{\geq 0}^{N \times N}$ such that $\text{rank}(M), \text{rank}_+^\varepsilon(M) \leq (\log N)^{C_\varepsilon}$, and $\text{rank}_+(M) > N^{c_\varepsilon}$.*

1.4 Applications in Communication Complexity

Analyzing lifted functions or pattern matrices has been a very useful tool in communication complexity over the last few years and our work builds on the techniques of [GLM⁺15] who show lifting theorems for various *rectangle-based* communication measures. Our main decomposition theorem, Theorem 2.10, can be used to recover the main results of [GLM⁺15]. Indeed, the main results of [GLM⁺15] follow from a structural result about approximating *rectangles* by *juntas* – an analogue of Theorem 2.7 that in turn follows easily from our decomposition theorem. For a more detailed comparison, see the discussion at the beginning of Section 6.3. We believe that our decomposition theorem could lead to other such applications in future.

²Note that $\deg_+(f + \eta) \leq \deg_+(f)$ for all $\eta > 0$.

2 Proof overview

2.1 Lifting \deg_+ lower bounds to non-negative rank

The proof of [Theorem 1.7](#) consists of two steps. First, we show that if $\text{rank}_+(M_f)$ is small for a function f , then f can be *approximated* by a conical junta under a carefully chosen notion of *approximation*. Second, we show that if f can be so approximated by a conical junta, then $\deg_+(f + \eta)$ is small for $\eta \ll 1$.

Towards making this outline more precise, we begin by defining a notion of approximate conical juntas that plays an important role in our proofs. We first state some basic notations that we use throughout:

- For any function f , $\mathbb{E}[f]$ denotes the expectation of f on the uniform distribution over its domain.
- For any $x \in \{-1, 1\}^n$ and $I \subseteq [n]$, we write x_I to denote the projection of x on to the coordinates in I .
- A *Boolean conjunction* $C : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ is defined by a subset $I \subseteq [n]$ of variables and α an assignment to the variables in I by $C(x) = 2^{|I|} \cdot \mathbf{1}[x_I = \alpha]$. We say C is a d -conjunction if $|I| \leq d$. Observe that we choose a non-standard scaling that satisfies $\mathbb{E}[C] = 1$.
- For any $S \subseteq [n]$, the parity function $\chi_S(x) = \prod_{i \in S} x_i$ for any $x \in \{-1, 1\}^n$. Any function $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ has the *Fourier expansion* $f(x) = \sum_{S \subseteq [n]} \widehat{f}(S) \chi_S(x)$. The terms $\widehat{f}(S)$ are the Fourier coefficients of f .

Definition 2.1 (ε -decaying functions). For $0 < \varepsilon < 1$, a function $h : \{-1, 1\}^n \rightarrow \mathbb{R}$ is said to be ε -decaying if $\mathbb{E}[h] = 0$ and for every $I \subseteq [n]$, $|\widehat{h}(I)| \leq \varepsilon^{|I|}$.

Definition 2.2 $((\varepsilon, \delta)$ -approximate conical d -junta). For $\varepsilon, \delta \in (0, 1)$, a function $f : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ with $\mathbb{E}[f] = 1$ is said to be an (ε, δ) -approximate conical d -junta if f can be written as

$$f(z) = \sum_{i \in [N]} \lambda_i C_i(z) \cdot (1 + h_i(z)) + \gamma(z) \quad (2.1)$$

for d -conjunctions C_1, \dots, C_N , ε -decaying functions h_1, \dots, h_N , $\lambda_1, \dots, \lambda_N \in \mathbb{R}_{\geq 0}$ with $\sum_i \lambda_i \leq 1$, and a function $\gamma : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathbb{E}[\gamma] \leq \delta$.

Notice that the approximation by conical-juntas has two distinct error terms, the multiplicative errors due to the ε -decaying functions $\{h_i\}$ and the additive error in the form of γ .

The first step in proving [Theorem 1.7](#) is the following lemma saying that if $\text{rank}_+(M_f)$ is small, then f is an approximate conical d -junta for small d .

Lemma 2.3 (Non-negative rank to Approximate Conical Juntas). *There exists a constant $\alpha_1 \geq 1$ such that the following holds. For $b \geq \alpha_1 \log n$, every function $f : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ with $\mathbb{E}[f] = 1$ is a $(2^{-b/2}, 2^{-bd/\alpha_1})$ -approximate conical d -junta for all $d \geq \alpha_1(\log \text{rank}_+(M_f^b))/b$.*

We defer the sketch of the proof of the lemma to the next section and continue with our outline of the proof of [Theorem 1.7](#).

Given the above lemma, the final step in proving [Theorem 1.7](#) is to show a connection between $\deg_+(f)$ and (ε, δ) -approximation by conical juntas. Specifically, we show a certain *robustness* of the class of conical juntas: if a function f is an (ε, δ) -approximate conical d -junta for sufficiently small ε and δ , then the function $f + \eta$ is an exact conical $8d$ -junta for a small constant η .

Lemma 2.4. *Suppose $f : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ with $E[f] < 1$ is a (ε, δ) -approximate conical d -junta for $\varepsilon < 1/n^4$ and $\delta < 1/n^{8d}$ then*

$$\deg_+\left(f + \frac{1}{n}\right) < 8d$$

[Lemma 2.3](#) and [Lemma 2.4](#) together imply [Theorem 1.7](#) almost immediately by setting the parameters appropriately; see [Section 4](#).

We defer the proof of [Lemma 2.4](#) to [Section 4.2](#). In what follows, we sketch the key ideas underlying the proof of [Lemma 2.3](#).

2.2 Approximating Rectangles by Conical Juntas

We now sketch the proof of [Lemma 2.3](#). To do so, we need the following basic definition³.

Definition 2.5 (Density). A function $p : [q]^n \rightarrow \mathbb{R}_{\geq 0}$ is said to be a density if $\mathbb{E}[p(x)] = 1$. A density p defines a corresponding random variable X on $[q]^n$ where $\Pr[X = x] = p(x) \cdot q^{-n}$. We denote $X \sim p$ this random variable.

Recall the statement of the lemma: we have a density f on $\{-1, 1\}^n$ such that M_f has small non-negative rank and we want to show that f is a low-degree approximate conical junta. Let $\text{rank}_+(M_f) = r$. By definition, $M_f = \sum_{i \in [r]} M_i$ where each M_i is a non-negative rank one matrix; further, by appropriate normalization, we can assume that $M_i = \lambda_i u_i u_i^\dagger$, where u_i, v_i are densities on $[q]^n$ and $\lambda_i > 0$. This decomposition of the matrix M_f into non-negative rank 1 matrices $\{M_i\}$, yields a corresponding decomposition of the function f into a sum of non-negative functions, one corresponding to each rank one matrix M_i .

Formally, let us denote by $G : [q]^n \times [q]^n \rightarrow \{-1, 1\}$ the function $G := g^{\otimes n}$. By definition, the entries of the matrix M_f are given by $M_f(x, y) = f(G(x, y))$. For $z \in \{-1, 1\}^n$, let $(X, Y) \sim G^{-1}(z)$ denote a uniformly random pair chosen from the set of pairs $G^{-1}(z) \subseteq [q]^n \times [q]^n$. With this notation,

$$f(z) = \mathbb{E}_{(X, Y) \sim G^{-1}(z)} M_f(X, Y) = \sum_{i \in [r]} \mathbb{E}_{(X, Y) \sim G^{-1}(z)} [M_i(X, Y)] = \sum_{i \in [r]} \lambda_i \mathbb{E}_{(X, Y) \sim G^{-1}(z)} [u_i(X) v_i(Y)] . \quad (2.2)$$

Borrowing terminology from communication complexity, we will refer to the rank one matrices $u_i v_i^\dagger$ as *rectangles*. In order to approximate the function f by a conical junta, it suffices to approximate the terms corresponding to each *rectangle* by a conical junta. We exhibit such an approximation for all *large rectangles*.

³We work with densities (instead of equivalently working with probability density functions or just non-negative functions) as keeping track of errors is cleaner under this normalization.

Towards this end, for two densities u, v on $[q]^n$, define $Acc_{u,v} : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ by

$$Acc_{u,v}(z) = \mathbb{E}_{(X,Y) \sim G^{-1}(z)} [u(X)v(Y)]. \quad (2.3)$$

Note that $\mathbb{E}[Acc] = \mathbb{E}[u(x)v(y)] = 1$. Thus, Acc is a density on $\{-1, 1\}^n$. Indeed, it is easy to check that $Acc_{u,v}$ is the density of the random variable $G(X, Y)$ for $X \sim u$ and $Y \sim v$ (X, Y independent). Using this definition in (2.2), we get

$$f(z) = \sum_{i=1}^r \lambda_i \cdot Acc_{u_i, v_i}(z).$$

This motivates the study of functions $Acc_{u,v}$ for rectangles. Indeed, structural results characterizing such functions form the core of previous results on pattern matrices [?, GLM⁺15]. We show that functions $Acc_{u,v}$ as above are *simple* when the rectangle $u \times v$ is *large*. To formalize this we need the notion of *min-entropy*.

Definition 2.6 (Min-Entropy). For a density u on $[q]^n$, the *min-entropy* of u , $H_\infty(u)$, is defined by⁴⁵

$$H_\infty(u) = \min_{x \in \{-1, 1\}^n} \log(q^n / u(x)).$$

For intuition, it is helpful to think of the special case where the densities u_i, v_i correspond to uniform distributions over some subsets U_i, V_i of $[q]^n$ respectively. The rectangle M_i is said to be *large*, if the sets U_i and V_i are both *large*, of size at least $q^n / 2^C$ for $C \ll n$. More generally, the rectangle M_i is *large* if the distributions u_i, v_i each have min-entropy at least $n \log q - C$. We will refer to C as the *min-entropy deficiency*.

Since M is the sum of r rectangles, one can argue that it is approximated by *large* rectangles whose min-entropy deficiency is at most $O(\log r)$. The contribution from all the *small* rectangles can be included into the additive error term $\gamma(z)$ in the approximation for f . The main work lies in showing that every *large rectangle* is approximated by conical juntas.

Theorem 2.7 (Junta Approximation). *There exists a constant $\alpha_2 \geq 1$ such that the following holds. Let u, v be densities over $[q]^n$ with $q = 2^b$ such that $H_\infty(u) + H_\infty(v) \geq 2b(n - t)$. Then, for all $b \geq \alpha_2 \log n$ and $d \geq \alpha_2 t$, $Acc_{u,v} : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ is a $(2^{-0.5b}, (2^{-0.5b})^d)$ -approximate conical d -junta.*

[GLM⁺15] also show a similar, but weaker, junta approximation theorem. In the present context, they essentially show that $Acc_{u,v}$ can be approximated as $Acc_{u,v}(z) = (1 \pm 2^{-\Omega(b)}) \cdot h(z) \pm 2^{-\Omega(bd)}$ where h is a conical d -junta. Note that the multiplicative error is only of the order $2^{-\Omega(b)}$ and this was a critical bottleneck in using their results to prove a lifting theorem for non-negative rank as in Theorem 1.7 (instead of for approximate non-negative rank). In comparison, we get exponentially small error in terms of approximate conical d -juntas. The latter is in fact stronger; a straightforward extension of our arguments can in fact recover the corresponding statement of [GLM⁺15].

⁴Note that this is the same as the more standard definition of $\min_{x \in \{-1, 1\}^n} \log(1 / \Pr[X = x])$ where $X \sim u$.

⁵Throughout this work, all logarithms are to the base 2.

2.3 Decomposing High-Entropy Distributions

The proof of [Theorem 2.7](#) relies on a crucial decomposition lemma for high-entropy distributions that may be of independent interest. Let u, v be two densities over $[q]^n$ with min-entropy at least $(n - C) \cdot \log q$ for some $C \ll n$ and let $X \sim u, Y \sim v$ be sampled independently.

A particularly simple class of high min-entropy distributions are those where a subset of C coordinates of X are fixed, while the rest are uniformly random. That is, for some set $I \subseteq [n]$ with $|I| \leq C$, X_I is a fixed string whereas $X_{[n] \setminus I}$ is uniformly random over $[q]^{[n] \setminus I}$. Similarly, Y could satisfy a similar property for a set $J \subseteq [n]$ with $|J| \leq C$. An especially desirable scenario is one where X, Y are *aligned* in the sense that $I = J$. For such aligned distributions, the random variable $Z = g^n(X, Y) \in \{-1, 1\}^n$ is such that Z_I is fixed while $Z_{\bar{I}}$ is uniformly random. In other words, the probability density of Z is a C -junta depending only on I .

We will show that as long as X, Y have high min-entropy, the product distribution $X \times Y$ can be decomposed into distributions that are essentially as simple and aligned as in the above discussion.

To this end, we next introduce the notion of *blockwise-dense* distributions; they were first defined in [\[GLM⁺15\]](#) and play a crucial role here.

Definition 2.8. A distribution X on $[q]^n$ is **BLOCKWISE-DENSE** if for every $I \subseteq [n]$, $H_\infty(X_I) \geq 0.8 \cdot \log q \cdot |I|$. We say a density u on $[q]^n$ is **BLOCKWISE-DENSE** if $X \sim u$ is **BLOCKWISE-DENSE**.

Definition 2.9. A distribution X on $[q]^n$ is a d -CBD (“conjunctive blockwise-dense”) distribution if for some set of coordinates $I \subseteq [n]$, $|I| \leq d$, $H_\infty(X_I) = 0$ and for every $J \subseteq [n] \setminus I$, $H_\infty(X_J) \geq 0.8 \cdot \log q \cdot |J|$. We refer to d as the *degree* of the CBD distribution, and the set of blocks I as the *fixed* blocks. We say two d -CBD distributions X, Y on $[q]^n$ are *aligned* if the *fixed* blocks I are the same in both.

Analogously, we say two densities u, v over $[q]^n$ are *aligned d -CBD* if the random variables X, Y are aligned d -CBD distributions for $X \sim u, Y \sim v$.

The technical core of our results is the following lemma stating that any two independent high-entropy densities u, v over $[q]^n$ can be approximated by a convex combination of aligned d -CBD densities for small d . The error of the approximation will depend on the entropy deficiency of $u \otimes v$ and the degree of the CBD distributions used in the approximation.

Theorem 2.10. *There exists a constant $c \geq 1$ such that the following holds. For $n \geq 1$ and $q \geq n^c$, let u, v be two densities on $[q]^n$ with $H_\infty(u) + H_\infty(v) \geq 2(n - t) \cdot \log q$. Then, for all $d \geq ct/(\log q)$, the product density $u \otimes v$ on $[q]^n \times [q]^n$ can be written as a convex combination of densities $u_1 \otimes v_1, u_2 \otimes v_2, \dots, u_N \otimes v_N$, and γ_{error} , i.e., $u \otimes v = \sum_{i=1}^N \lambda_i u_i \otimes v_i + \lambda_{\text{error}} \gamma_{\text{error}}$, such that*

- $0 \leq \lambda_1, \dots, \lambda_N, \lambda_{\text{err}} \leq 1, \sum_{i=1}^N \lambda_i + \lambda_{\text{err}} = 1$.
- $|\lambda_{\text{err}}| < q^{-\Omega(d)}$.
- For every $i \in [N]$, $X_i \sim u_i, Y_i \sim v_i$ are aligned d -CBD distributions.

[Theorem 2.7](#) follows easily from the above using some “extractor”-like properties (cf. [Fact 5.2](#)) of the inner-product function g . We defer the details of the proof of the theorem to the corresponding section.

2.4 Organization

We present the proof in a top-down manner: We first prove [Theorem 1.7](#) assuming [Theorem 2.7](#). We then prove [Theorem 2.7](#) assuming [Theorem 2.10](#) (this is almost immediate). Finally, we prove [Theorem 2.10](#). We then prove [Theorem 1.2](#), [Corollary 1.3](#) in [Section 7](#).

3 Preliminaries

We describe some basic notation that we use throughout⁶.

3.1 Basic Notation

1. \mathcal{P}_d^n denotes the collection of all polynomials of degree at most d on n variables on $\{-1, 1\}^n$.
2. $\mathbf{1}(E)$ is the indicator for the event E normalized to have mean 1. That is, $\mathbf{1}(E)$ is 0 when E doesn't happen and $1/\mathbb{P}[E]$ when E happens.
3. For any function f , $\mathbb{E}[f]$ denotes the expectation of f on the uniform distribution over its domain.
4. For matrices M , $\mathbb{E}[M]$ denotes the expectation of $M(x, y)$ under x, y being uniformly random indices for its rows and columns.
5. For any $x \in \{-1, 1\}^n$ and $I \subseteq [n]$, we write x_I to denote the projection of x on to the coordinates in I .
6. A *Boolean conjunction* $C : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ is defined by a subset $I \subseteq [n]$ of variables and α an assignment to the variables in I by $C(x) = 2^{|I|} \cdot \mathbf{1}[x_I = \alpha]$. We say C is a d -conjunction if $|I| \leq d$. Observe that we choose a non-standard scaling that satisfies $\mathbb{E}[C] = 1$.
7. For any $S \subseteq [n]$, the parity function $\chi_S(x) = \prod_{i \in S} x_i$ for any $x \in \{-1, 1\}^n$. Any function $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ has a Fourier expansion: $f(x) = \sum_{S \subseteq [n]} \widehat{f}(S) \chi_S(x)$. The terms $\widehat{f}(S)$ are the Fourier coefficients of f .

3.2 Sherali-Adams Linear Programming Relaxations

Our results relate arbitrary linear programming relaxations for CSPs to the Sherali-Adams hierarchy. We discuss the latter class of linear programs next. We begin with the definition of a *degree d pseudo-expectation*

Definition 3.1 (Sherali-Adams Pseudoexpectation). A degree d Sherali-Adams pseudoexpectation, \mathbb{E} , is a linear operator on the space of degree at most d polynomials, \mathcal{P}_d^n , such that

1. For every non-negative $p \in \mathcal{P}_d^n$ that depends on only d variables, $\mathbb{E}[p] \geq 0$, and

⁶To have all notations together, some are repeated from the introduction.

2. $\tilde{\mathbb{E}}[1] = 1$.

Since $\tilde{\mathbb{E}}$ is a linear, it is completely specified by its values on multilinear polynomials, in particular by the values $\tilde{\mathbb{E}}[\chi_S(x)]$ for $S \subseteq [n]$, $|S| \leq d$.

Sherali-Adams linear programming relaxations can be equivalently described using a collection of probability distributions over local assignments. The above view is more convenient for us. We refer the reader to [CLRS13] for a detailed discussion.

The degree d -Sherali-Adams linear programming relaxation for a CSP solves the following optimization problem. Given an instance I of a k -ary CSP, we can canonically encode it as a polynomial of degree k $P_I : \{-1, 1\}^n \rightarrow [0, 1]$ such that $P_I(x) = I(x)$ for all assignments $x \in \{-1, 1\}^n$. Then, the degree d -Sherali-Adams relaxation is

$$\max_{\tilde{\mathbb{E}}} \tilde{\mathbb{E}}[P_I(x)], \quad (3.1)$$

where $\tilde{\mathbb{E}}$ ranges over all degree d Sherali-Adams pseudoexpectations. We define $\text{SA}_d(I)$ as the value of the the optimization problem (3.1). The above optimization problem can be solved using a linear program on $n^{O(d)}$ variables and constraints. Note that $\text{opt}(I) \leq \text{SA}_d(I)$.

Sherali-Adams LP and Non-negative Degree: Linear programming duality gives an elegant characterization of the performance of Sherali-Adams LP on a CSP in terms of non-negative degree.

Fact 3.2 (Sherali-Adams value and Non-negative Degree [CLRS13]). *Let $P : \{-1, 1\}^k \rightarrow \{0, 1\}$ be a predicate and I be an instance of $\text{CSP}(P)$. Then, $\text{SA}_d(I) \leq c$ if and only if $\deg_+(c - I) \leq d$.*

4 Juntas, Rectangles, and Non-negative Rank of Lifted Matrices

In this section, we will show our main Theorem 1.7 assuming Theorem 2.7. Let $f : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ be as in the theorem with $\mathbb{E}[f] = 1$, and let the gadget $g : [q] \times [q] \rightarrow \{-1, 1\}$ and b be as in the theorem. We will show that if $\text{rank}_+(M_f)$ is small, then $\deg_+(f + \eta)$ is small, where $\eta = O(1/n)$. Concretely, given a small rank non-negative factorization of M_f , we use the factorization to get a small-degree conical junta approximating $f + \eta$. As described in the introduction, this is done in two modular steps: Lemmas 2.3, and 2.4. First, we show how Lemma 2.3 and Lemma 2.4 together immediately imply Theorem 1.7.

Proof of Theorem 1.7. Fix a constant C such that $C \geq \max(16\alpha_1, 1000)$ for α_1 from Lemma 2.3. Let $R := \text{rank}_+(M_f^b)$. By Lemma 2.3, this implies that f is a $(2^{-b/2}, 2^{-bd/\alpha_1})$ -approximate conical d -junta for $d \geq \alpha_1 \log R/b$. For $b \geq C \log n$, $2^{-b} \leq \min(1/n^{1000}, 1/n^{16\alpha_1})$. Hence, f is a $(1/n^{500}, 1/n^{16d})$ -approximate conical d -junta with $d = \lceil \alpha_1 \log R/b \rceil$. By Lemma 2.4, this implies that

$$\deg_+\left(f + \frac{1}{n}\right) \leq 8 \cdot \frac{\alpha_1 \log R}{b},$$

which yields the inequality,

$$R \geq 2^{\Omega(b \cdot \deg_+(f + 1/n))}.$$

□

For the rest of this section we adopt the following assumptions:

Important Parameters.

- $f : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ with $\mathbb{E}[f] = 1$.
- The *block-length* of the gadget $b = C \log n$ for a sufficiently large constant C . (Recall that the gadget is the inner-product function).

4.1 Approximation by Approximate Conical Juntas

Here we prove [Lemma 2.3](#) which we restate for convenience.

Lemma (Restatement of [Lemma 2.3](#)). *There exists a constant $\alpha_1 \geq 1$ such that the following holds. For $b \geq \alpha_1 \log n$, every function $f : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ with $\mathbb{E}[f] = 1$ is a $(2^{-b/2}, 2^{-bd/\alpha_1})$ -approximate conical d -junta for all $d \geq \alpha_1(\log \text{rank}_+(M_f^b))/b$.*

Proof. For the sake of brevity, let us set $G := \mathcal{G}^{\otimes n}$ and $R := \text{rank}_+(M_f^b)$. For every $z \in \{-1, 1\}^n$, and $(x, y) \in G^{-1}(z)$, we have $M_f(x, y) = f(z)$. The high-level idea is as follows. From the definition of M_f , we have

$$f(z) = \mathbb{E}_{(X,Y) \sim G^{-1}(z)} [M_f(X, Y)].$$

Further, by definition of $\text{rank}_+(M_f)$, the matrix M_f can be expressed as a sum of $\text{rank}_+(M_f)$ non-negative rank-1 matrices. In turn, this yields a decomposition of f into a sum of a family of non-negative functions. We then use [Theorem 2.7](#) to approximate each of these functions by approximate conical juntas, thereby yielding the desired approximation for f .

Concretely, from the definition of non-negative rank, there exists a collection of densities on $\{-1, 1\}^{bn}$, $\{u_i \mid 1 \leq i \leq R\}$ and $\{v_i \mid 1 \leq i \leq R\}$, and a set of non-negative constants $\lambda_1, \lambda_2, \dots, \lambda_R$ such that

$$M_f = \sum_{i=1}^R \lambda_i u_i v_i^\dagger.$$

Observe that

$$\sum_{i=1}^R \lambda_i = \sum_{i=1}^R \lambda_i \mathbb{E}[u_i v_i^\dagger] = \mathbb{E}[M_f] = 1.$$

Now, for any $z \in \{-1, 1\}^n$,

$$\begin{aligned} f(z) &= \mathbb{E}_{(X,Y) \in G^{-1}(z)} [M_f(X, Y)] \\ &= \mathbb{E}_{(X,Y) \in G^{-1}(z)} \left[\sum_{i=1}^R \lambda_i u_i(X) v_i(Y) \right] \\ &= \sum_{i=1}^R \lambda_i \cdot \mathbb{E}_{(X,Y) \in G^{-1}(z)} [u_i(X) v_i(Y)] \end{aligned}$$

$$= \sum_{i=1}^R \lambda_i \cdot \text{Acc}_{u_i, v_i}(z).$$

(Recall the definition of Acc from Equation (2.3).)

In analogy with communication complexity, we will refer to the rank 1 matrices $u_i v_i^\dagger$ as *rectangles*. We will split the family of rectangles in to *large* and *small*. To this end, fix $t = 4 \log(R)$. A rectangle $u_i v_i^\dagger$ will be referred to as *large*, if the min-entropies of u_i and v_i are large. More precisely, let

$$Q = \{i \in R \mid H_\infty(u_i) + H_\infty(v_i) \geq 2(n-t)(\log q)\}.$$

We can now write f as a sum $f = J + \delta_1$ where,

$$J(z) = \sum_{i \in Q} \lambda_i \text{Acc}_{u_i, v_i}(z)$$

and

$$\delta_1(z) = \sum_{i \notin Q} \lambda_i \text{Acc}_{u_i, v_i}(z).$$

Now, for each $i \in Q$, by Theorem 2.7 applied to u_i, v_i , Acc_{u_i, v_i} is a $(\varepsilon, \varepsilon^d)$ -approximate conical d -junta for all $d \geq \alpha_2 t$. Therefore, J is an (ε, δ') -approximate conical d -junta with $\delta' = \left(\sum_{i \in Q} \lambda_i\right) \cdot \varepsilon^d \leq \varepsilon^d$.

Now we will bound the total additive error due to the small rectangles. Observe that for any $i \notin Q$, $\lambda_i \leq 2^{-t/2}$. This is because, $\lambda_i \mathbb{E}_y[u_i(x)v_i(y)] = \lambda_i u_i(x) \leq \mathbb{E}_y[M_f(x, y)] = \mathbb{E}[f] = 1$ and similarly, $\lambda_i v_i(y) \leq 1$ for any y . Further, recall that $\text{Acc}_{u, v}$ is a density for all densities u, v . Thus,

$$\mathbb{E}\left[\sum_{i \notin Q} \lambda_i \text{Acc}_{u_i, v_i}\right] \leq \sum_{i \notin Q} 2^{-t/2} \leq 2^{-t/2} R.$$

Therefore, $f = J + \delta_1$ is an $(\varepsilon, \varepsilon^d + 2^{-t/2} R)$ -approximate conical d -junta for all $d \geq \alpha_2 t/b$. Choosing $t = 4 \log R$ and $d = \max\{4\alpha_2 \log R/b, 2 \log R/b\}$ proves the lemma. \square

4.2 Approximate Conical Juntas to Conical Juntas

Notation.

- $C_{\leq D}$: cone of non-negative D -juntas on $\{-1, 1\}^n$.
- $\mathcal{L} : \{-1, 1\}^n \rightarrow \mathbb{R}$: separating function.

In this section we will prove Lemma 2.4 which asserts that if f is a low-degree approximate conical junta, then $f + \eta$ is a low-degree conical junta for η sufficiently small.

Lemma (Restatement of Lemma 2.4). *Suppose $f : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ with $E[f] < 1$ is a (ε, δ) -approximate conical d -junta for $\varepsilon < 1/n^4$ and $\delta < 1/n^{8d}$ then*

$$\deg_+\left(f + \frac{1}{n}\right) < 8d$$

At a high-level the proof is as follows. Suppose for the sake of contradiction that $\deg_+(f + 1/n) \geq 8d$. Then, there is a *nice separating* functional \mathcal{L} such that $\langle \mathcal{L}, f \rangle \leq -1/n$, and $\langle \mathcal{L}, h \rangle \geq 0$ for all $h \in C_{\leq 8d}$. We then use further properties of the functional that the latter property implies $\langle \mathcal{L}, h \rangle > -1/n$ for all $(\varepsilon, \varepsilon^d)$ -approximate conical d -junta - leading to a contradiction.

We first develop the requisite technical machinery concerning conical juntas and separating functionals.

Lemma 4.1. *Suppose $D < \deg_+(f + \eta)$. There exists a degree D function $\mathcal{L} : \{-1, 1\}^n \rightarrow \mathbb{R}$ such that:*

1. $\mathbb{E}[\mathcal{L}] = \mathbb{E}[\mathcal{L} \cdot 1] = 1$.
2. $\mathbb{E}[\mathcal{L}f] < -\eta$.
3. $\mathbb{E}[\mathcal{L}h] \geq 0$ for every conical D -junta h on $\{-1, 1\}^n$.
4. $|\widehat{\mathcal{L}}(S)| \leq 1$ for every $|S| \leq D$.
5. $\|\mathcal{L}\|_\infty \leq n^D$

Proof. Observe that the set of conical $\leq D$ -juntas denoted by $C_{\leq D}$ is convex. On the other hand from the hypothesis, we have that $f + \eta \notin C$. Thus, there exists a function $\mathcal{L} : \{-1, 1\}^n \rightarrow \mathbb{R}$ such that $\langle \mathcal{L}, h \rangle \geq 0$ for every $h \in C_{\leq d}$ but $\langle \mathcal{L}, (f + \eta) \rangle < 0$. Moreover, since $C_{\leq D}$ is contained in the linear subspace of degree D polynomials, without loss of generality, we can assume that \mathcal{L} is also a degree D polynomial.

The first three properties are simple to verify. Since the constant function $\mathbf{1} \in C_{\leq D}$, we can assume (by rescaling, if needed) that $\langle \mathcal{L}, \mathbf{1} \rangle = 1$ giving us the first property. Further, since $\langle \mathcal{L}, f \rangle = \mathcal{L} \cdot (f + \eta) - \langle \mathcal{L}, \eta \rangle \leq -\eta$ giving us the second property. The third property follows from our definition of \mathcal{L} .

We next bound the Fourier coefficients of \mathcal{L} . First observe that for any $S \subseteq [n]$, $|S| \leq D$, $\mathbf{1} + \chi_S$ is a non-negative D -junta. Therefore, $\langle \mathcal{L}, \mathbf{1} + \chi_S \rangle \geq 0$ so that $\langle \mathcal{L}, \chi_S \rangle \geq -\langle \mathcal{L}, \mathbf{1} \rangle = -1$. Similarly, $\langle \mathcal{L}, \mathbf{1} - \chi_S \rangle \geq 0$ so that $\langle \mathcal{L}, \chi_S \rangle \leq 1$. Thus, $|\widehat{\mathcal{L}}(S)| = |\langle \mathcal{L}, \chi_S \rangle| \leq 1$.

The final property follows as $\|\mathcal{L}\|_\infty = \|\sum_{S, |S| \leq D} \widehat{\mathcal{L}}(S) \chi_S(x)\| \leq \sum_{|S| \leq D} \|\widehat{\mathcal{L}}(S)\| \leq n^D$. \square

The following technical property of \mathcal{L} constructed in Lemma 4.1 will be required in our proof.

Lemma 4.2. *Let $\mathcal{L} : \{-1, 1\}^n \rightarrow \mathbb{R}$ be the separating function of degree D given by Lemma 4.1. Let $h : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ be a non-negative junta that depends only on variables $T \subseteq [n]$ and let S be any subset of $[n]$ such that $|S| + |T| \leq D$. Then,*

$$\mathbb{E}[\mathcal{L}h\chi_S] \leq \mathbb{E}[\mathcal{L}h].$$

Proof. Note that $h(1 - \chi_S)$ is a non-negative D -junta. Therefore, $\langle \mathcal{L}, h(1 - \chi_S) \rangle \geq 0$; the claim follows. \square

We are now ready to prove Lemma 4.3 which can be seen as a robust version of the property that $\mathbb{E}[\mathcal{L}h] \geq 0$ for every non-negative D -junta h .

Lemma 4.3. Let \mathcal{L} be a separating function of degree $D = 4d$ as in Lemma 4.1. Then, for any non-negative d -junta c with $\|c\|_\infty \leq 1$, and any $(1/n^4)$ -decaying function h , $\mathbb{E}[\mathcal{L}c(1+h)] \geq -n^{-8d}$.

Proof. Write $h = h_{low} + h_{high}$ where $h_{low} = \sum_{|S| \leq D-d} \widehat{h}(S) \chi_S$, and $h_{high} = \sum_{|S| > D-d} \widehat{h}(S) \chi_S$.

We have:

$$\mathbb{E}[\mathcal{L}c(1+h)] = \mathbb{E}[\mathcal{L}c(1+h_{low})] + \mathbb{E}[\mathcal{L}ch_{high}].$$

Let $\varepsilon = 1/n^4$. Now, $\mathcal{L}c$ is of degree at most $(D+d)$ and $\|\mathcal{L}c\|_\infty \leq n^D$. Further, for any x ,

$$\begin{aligned} |h_{high}(x)| &= \left| \sum_{S: |S| \geq D-d} \widehat{h}(S) \chi_S(x) \right| \leq \sum_{S: |S| \geq D-d} \varepsilon^{|S|} \\ &\leq \sum_{\ell=D-d}^n \varepsilon^\ell \cdot n^\ell \\ &\leq 2(\varepsilon n)^{D-d}, \end{aligned}$$

where the last inequality follows as $\varepsilon n = 1/n^3 < 1/2$. Therefore, $\|h_{high}\|_\infty \leq 2(\varepsilon n)^{D-d}$ and

$$|\mathbb{E}[\mathcal{L}ch_{high}]| \leq \|\mathcal{L}c\|_\infty \cdot \|h_{high}\|_\infty \leq 2n^{2D-d} \cdot \varepsilon^{D-d}. \quad (4.1)$$

Next, note that h_{low} is a linear combination of parities of degree at most d and that c is a function of degree at most d . Thus, by Lemma 4.2, we have,

$$\begin{aligned} |\mathbb{E}[\mathcal{L}ch_{low}]| &\leq \sum_{|S| \leq D-d} |\widehat{h}(S)| \mathbb{E}[\mathcal{L}c \chi_S] \\ &\leq \sum_{|S| \leq D-d} |\widehat{h}(S)| \mathbb{E}[\mathcal{L}c]. \end{aligned}$$

Since by definition $\mathbb{E}[h] = 0$, we have:

$$\mathbb{E}[\mathcal{L}c(1+h_{low})] \geq \mathbb{E}[\mathcal{L}c](1 - \sum_{1 \leq |S| \leq D-d} \varepsilon^{|S|}) \geq \mathbb{E}[\mathcal{L}c](1 - 2Dn\varepsilon). \quad (4.2)$$

Using that $\varepsilon < 1/n^4$ and $D < n$, we have $\mathbb{E}[\mathcal{L}c(1+h_{low})] \geq 0$. Using (4.1) and (4.2),

$$\mathbb{E}[\mathcal{L}c(1+h)] \geq -2\varepsilon^{3d} n^{7d} \geq -n^{-8d}.$$

□

Finally, we can complete the proof of Lemma 2.4.

Proof of Lemma 2.4. Fix $\eta = \frac{1}{n}$. For the sake of contradiction, assume that $\deg_+(f + \eta) \geq 8d$. Consider the functional \mathcal{L} given by Lemma 4.1 with $D = 4d$.

Since f is an (ε, δ) -approximate d -junta, we have

$$f(z) = \sum_i \lambda_i c_i(z)(1 + h_i(z)) + \gamma(z)$$

where the functions h_i are ε -decaying and the function γ satisfies $\mathbb{E}[|\gamma|] \leq \delta$. Now, apply the linear functional \mathcal{L} to both sides of the above equation. On one side, we get

$$\mathbb{E}[\mathcal{L} \cdot f] \leq -\eta.$$

On the other side, we get

$$\begin{aligned} \sum_i \lambda_i \mathbb{E}[\mathcal{L} c_i(1 + h_i)] + \mathbb{E}[\mathcal{L} \gamma] &\geq -\left(\sum_i \lambda_i\right) \cdot \frac{1}{n^{8d}} - \|\mathcal{L}\|_\infty \mathbb{E}[|\gamma|] \\ &\geq -\frac{1}{n^{8d}} - n^{4d} n^{-8d} > -\eta, \end{aligned}$$

yielding a contradiction. \square

5 The Junta Approximation Theorem

Here we prove [Theorem 2.7](#) assuming [Theorem 2.10](#). In addition to the latter decomposition, the proof relies on certain *extractor* properties of the inner-product function. Concretely, we need the following statement about the distribution of $G(X, Y)$ for blockwise-dense random variables X, Y that is implicit in [\[GLM⁺15\]](#).

Lemma 5.1. *Fix $q = 2^b$ for $b > 7$ and identify $[q]$ with $\{-1, 1\}^b$. Let $g : \{0, 1\}^b \times \{0, 1\}^b \rightarrow \{-1, 1\}$ be the Boolean inner product function.*

Suppose X and Y are independent, blockwise-dense random variables on $[q]^n$. Let ν be the density of the random variable $g^{\otimes n}(X, Y)$ on $\{-1, 1\}^n$. Then, $\nu = 1 + h$ for an ε -decaying function h where $\varepsilon = 2^{-0.5b}$.

The lemma is an easy consequence of the fact that the inner-product function is a *two-source extractor* for sufficiently high-entropies:

Fact 5.2 (Chor-Goldreich [\[CG88\]](#)). *Suppose X, Y are independent random variables over $\{-1, 1\}^\ell$ for $\ell > 7$ with min-entropy $H_\infty(X), H_\infty(Y) \geq 0.8\ell$. Then,*

$$|\mathbb{E}[g(X, Y)]| \leq 2^{-0.6\ell+1}.$$

Proof of Lemma 5.1. By [Fact 5.2](#), for any $S \subseteq [n]$ we have

$$\widehat{\nu}(S) = \mathbb{E}[\nu(z) \chi_S(z)] = \mathbb{E}_{z \sim \nu}[\chi_S(z)] = \mathbb{E}[\prod_{i \in I} g(X_{\{i\}}, Y_{\{i\}})] = \mathbb{E}[g(X_S, Y_S)].$$

Thus, $|\widehat{\nu}(S)| \leq 2^{-0.6b|S|+1} < 2^{-0.5b}$.

Let $h(z) = \sum_{|S| \geq 1} \widehat{\nu}(S) \chi_S(z)$. Then, $\nu = 1 + h$ and by the above estimate, h is ε -decaying for $\varepsilon = 2^{-0.5b}$. \square

Lemma 5.1 showed that if X, Y are BLOCKWISE-DENSE random variables, then the density of $g^{\otimes n}(X, Y)$ is an ε -decaying perturbation of the uniform density. In the following, we show a refinement of Lemma 5.1 when X, Y are aligned d -CBD random variables - specifically, that $g^{\otimes n}(X, Y)$ is an ε -decaying perturbation of a non-negative d -junta.

Lemma 5.3. *Let X, Y be aligned d -CBD random variables over $(\{0, 1\}^b)^n$ for $b > 7$ with the aligned blocks $I \subseteq [n]$ and $X_I = \alpha, Y_I = \beta$. Let v be the density of $z = g^{\otimes n}(X, Y)$. Then, for $\varepsilon = 2^{-0.5b}$, there exists an ε -decaying function h such that*

$$v = 2^{|I|} \cdot \mathbf{1}[z_I = g^{\otimes |I|}(\alpha, \beta)] \cdot (1 + h).$$

In particular, if u, v are aligned d -CBD densities over $(\{-1, 1\}^b)^n$, then $\text{Acc}_{u,v}$ is a $(2^{-.5b}, 0)$ -approximate conical d -junta.

Proof. Let $Z = g^{\otimes n}(X, Y)$. Then, $Z_I = g^{\otimes I}(\alpha, \beta)$ and $Z_{\bar{I}} = g^{\otimes \bar{I}}(X_{\bar{I}}, Y_{\bar{I}})$. In particular, the density of Z can be written as $v_I \cdot v_{\bar{I}}$ where v_I is the density of Z_I and $v_{\bar{I}}$ the density of $Z_{\bar{I}}$.

Now, by definition, $X_{\bar{I}}, Y_{\bar{I}}$ are d -CBD random variables. Thus, by [Lemma 5.1](#), the density $v_{\bar{I}}$ of $Z_{\bar{I}}$ can be written as $1 + h$ for a $2^{-0.5b}$ -decaying function. This completes the proof of the first part of the statement. The next part follows from the definition of $\text{Acc}_{u,v}$. \square

We are now ready to prove [Theorem 2.7](#) which we restate for convenience.

Theorem (Restatement of [Theorem 2.7](#)). *There exists a constant $\alpha_2 \geq 1$ such that the following holds. Let u, v be densities over $[q]^n$ with $q = 2^b$ such that $H_\infty(u) + H_\infty(v) \geq 2b(n - t)$. Then, for all $b \geq \alpha_2 \log n$ and $d \geq \alpha_2 t$, $\text{Acc}_{u,v} \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ is a $(2^{-0.5b}, (2^{-0.5b})^d)$ -approximate conical d -junta.*

Proof. We apply [Theorem 2.10](#) to u, v to write

$$u \otimes v = \sum_{i=1}^N \lambda_i u_i \otimes v_i + \lambda_{\text{error}} \gamma_{\text{error}},$$

as guaranteed by the theorem. Then,

$$\text{Acc}_{u,v} = \sum_{i=1}^N \lambda_i \text{Acc}_{u_i, v_i} + \lambda_{\text{error}} \gamma'(z),$$

where γ denotes the distribution of $G(X, Y)$ for $(X, Y) \sim D_{\text{error}}$. Now, as u_i, v_i are aligned d -CBD densities, by [Lemma 5.3](#) each Acc_{u_i, v_i} is a $(2^{-.5b}, 0)$ -approximate conical d -junta. Hence, $\text{Acc}_{u,v}$ is a $(2^{-.5b}, \lambda_{\text{err}})$ -approximate conical d -junta. The claim now follows from the bound on λ_{err} . \square

6 Decomposition of High Min-Entropy Distributions

In this section we prove [Theorem 2.10](#). In fact, we show a stronger decomposition theorem that is no more difficult to prove and is needed to recover the results of [\[GLM⁺15\]](#) in our framework. We will use the following notation:

- We use distributions, densities and random variables interchangeably with the meaning being clear from the context.

- For μ a density on some domain \mathcal{D} and $S \subseteq \mathcal{D}$, we write $\mu|_S$ for the density μ conditioned on S . We also define $\mu(S) = \Pr_{X \sim \mu}[X \in S]$.
- For a random variable X on $[q]^n$ and $I \subseteq [n]$, we write X_I to denote X projected to the coordinates in I . For a density μ on $[q]^n \times [q]^n$ and $I \subseteq [n]$, we write μ_I to denote the density on $[q]^{|I|} \times [q]^{|I|}$ obtained by projecting μ to the coordinates in I .
- For brevity, we say a density μ on $[q]^n \times [q]^n$ is an aligned d -CBD if its two marginals along $[q]^n$ are aligned d -CBD densities.

6.1 Warm Up: One-Dimensional Decompositions

Observe that by definition, any d -CBD density has min-entropy at least $0.8 \log q(n - d)$ which for $d \ll n$, we consider high. Thus, any convex combination of d -CBD densities also has high min-entropy. One could ask for a converse at this point: can every high min-entropy density be written as a convex combination of d -CBD densities for small d ? As a warmup for the more general decomposition, we first show that this is indeed the case.

Lemma 6.1. *Let μ is a density on $[q]^n$ with $H_\infty(\mu) \geq (n - t) \cdot \log q$. Then, there exists a partition of $[q]^n$ as*

$$[q]^n = \left(\bigcup_{i \in [N]} S_i \right) \cup S_{\text{error}}$$

such that

- For each $i \in [N]$, $\mu|_{S_i}$ is a $10t$ -CBD distribution.
- $\mu(S_{\text{error}}) \leq q^{-t}$.

Proof. We present an algorithm that obtains the claimed partition.

Setup: μ is a distribution over $[q]^n$ with $H_\infty(\mu) \geq (n - t) \cdot \log q$.

Decompose(S)

Input $S \subseteq [q]^n$.

1. If $\mu|_S$ is BLOCKWISE-DENSE, TERMINATE and return S .
2. If $\mu(S) \leq q^{-t}$, TERMINATE and return S , labeled as S_{error} .
3. Else, let $Y \sim \mu|_S$. Let $I \subseteq [n]$ be a maximal set such that $H_\infty(Y_I) < 0.8 \log q \cdot |I|$ and suppose $\alpha \in [q]^I$ be such that $p = \Pr_{\mu|_S}[Y_I = \alpha] \geq q^{-0.8|I|}$. Then, set $S_1 \leftarrow S \cap \{y \in [q]^n : y_I = \alpha\}$ and $S_2 \leftarrow S \cap \{y \in [q]^n : y_I \neq \alpha\}$.
4. RETURN $S_1 \cup \text{DECOMPOSE}(S_2)$.

To get the desired decomposition of μ , we call $\text{DECOMPOSE}([q]^n)$. Before we analyze the decomposition so produced, let us consider a single execution of the subroutine. Consider an execution of $\text{Decompose}(S)$ that terminates in Step 4, returning a subset S_1 and calling $\text{Decompose}(S_2)$. We make the following observations.

Claim 6.2. $\mu_{|S_1}$ is blockwise-dense except for the fixed coordinates in I .

Proof. Since I is a maximal set such that $H_\infty(Y_I) < 0.8 \log q |I|$, for any subset $J \subseteq [n]/I$, we have $H_\infty(Y_J | Y_I = \alpha) > 0.8 \log q |J|$. \square

Claim 6.3. $|I| \leq 10t$

Proof. For any $\beta \in [q]^{|I|}$,

$$\begin{aligned} \mathbb{P}_{\mu_{|S}}[Y_I = \beta] &= \frac{\mathbb{P}_\mu[Y_I = \beta]}{\mu(S)} \\ &\leq \frac{\mathbb{P}_\mu[Y_I = \beta]}{q^{-t}} && \text{(Step(2) did not terminate)} \\ &\leq \frac{q^{t-|I|}}{q^{-t}} = q^{2t-|I|} && (H_\infty(\mu) \geq (n-t)(\log q)) \end{aligned}$$

Since there exists $\alpha \in [q]^I$ such that $\mathbb{P}_{\mu_{|S}}[Y_I = \alpha] \geq q^{-0.8|I|}$, we get that $q^{-0.8|I|} \leq q^{2t-|I|}$. This implies that $|I| \leq 10t$. \square

From the above claims, it is clear that $\mu_{|S_1}$ is a $10t$ -CBD distribution, whenever $\text{Decompose}(S)$ terminates in Step (4).

Suppose we call $\text{Decompose}([q]^n)$, the recursive algorithm will return a partition of $[q]^n$ into subsets $\{S_i\}_{i \in [N]}$ and eventually terminate either via Step(1) or Step(2). If the algorithm terminates via Step (1), then $\mu_{|S_i}$ is $10t$ -CBD for all the sets S_i and the lemma follows. If the algorithm terminates via Step (2), then it produces a subset S_{error} with $\mu(S_{\text{error}}) \leq q^{-t}$, as desired. \square

6.2 Rectangular Decompositions

To prove our junta theorem, [Theorem 2.7](#), and for other plausible applications in communication complexity, the decomposition obtained by [Lemma 6.1](#) does not suffice. In particular, the underlying domain is two-dimensional $[q]^n \times [q]^n$, and the partitions need to be *rectangular*. In this section, we will prove a general *rectangular decomposition* theorem designed for distributions with high min-entropy.

A combinatorial rectangle $\mathcal{R} \subseteq [q]^n \times [q]^n$ is given by $\mathcal{R} = \mathcal{A} \times \mathcal{B}$ for $\mathcal{A}, \mathcal{B} \subseteq [q]^n$. We can now state our main decomposition theorem.

Theorem 6.4 (Rectangular Decompositions). *Let μ be a probability density on $[q]^n \times [q]^n$ such that for all $I \subseteq [n]$,*

$$H_\infty(\mu_I) \geq 1.9 \cdot \log q \cdot |I| - t.$$

Then for all $d \in \mathbb{N}$, there exists a partition

$$[q]^n \times [q]^n = \left(\bigcup_{i \in [N]} \mathcal{R}_i \right) \cup \text{Error}$$

where $\{\mathcal{R}_i = \mathcal{A}_i \times \mathcal{B}_i\}_{i \in [N]}$ are rectangles such that

1. For each $i \in [N]$, $(X_i, Y_i) \sim \mu|_{\mathcal{R}_i}$, X_i, Y_i are aligned d -CBD.
2. $\mu(\text{Error}) \leq 2^t \cdot (dq^{-0.05})^d$

Notice that the above theorem also implies that the density μ can be approximated by a convex combination of aligned d -CBD distributions by setting,

$$\mu = \sum_{i \in [N]} \mu(\mathcal{R}_i) \cdot \mu|_{\mathcal{R}_i} + \mu(\text{Error}) \cdot \mu|_{\text{Error}}.$$

[Theorem 2.10](#) is an immediate consequence of the above decomposition theorem.

Proof of [Theorem 2.10](#). Let μ be the density $u \otimes v$ on $[q]^n \times [q]^n$. Then, μ for all $I \subseteq [n]$, $H_\infty(\mu_I) \geq 2(\log q)|I| - t$. Now, apply [Theorem 6.4](#) to obtain a partition of $[q]^n \times [q]^n$ as $\bigcup_{i \in [N]} \mathcal{R}_i \cup \text{Error}$ satisfying the conditions of the theorem. Then,

$$\mu = \sum_{i=1}^N \mu(\mathcal{R}_i) \cdot \mu|_{\mathcal{R}_i} + \mu(\text{Error})\mu|_{\text{Error}}.$$

Note that $\sum_{i=1}^N \mu(\mathcal{R}_i) + \mu(\text{Error}) = 1$. Let $\mathcal{R}_i = \mathcal{A}_i \times \mathcal{B}_i$. Then, we can write $\mu|_{\mathcal{R}_i} = u_i \otimes v_i$, where $u_i = u|_{\mathcal{A}_i}$ and $v_i = v|_{\mathcal{B}_i}$. Then, $X_i \sim u_i, Y_i \sim v_i$ are aligned d -CBD and

$$\mu(\text{Error}) \leq 2^t (dq^{-0.05})^d \leq 2^{-t} (nq^{-0.05})^d \leq 2^{-t} q^{-\Omega(d)} \leq q^{-\Omega(d)},$$

for $q \geq n^c, d \geq ct$ for a sufficiently big constant $c \geq 1$. This proves the theorem. \square

6.3 The Decomposition Algorithm

We will show [Theorem 6.4](#) by devising an algorithm that constructs the partition given the distribution μ and a parameter $d \in \mathbb{N}$. The algorithm is a natural extension of the one used in the proof of [Lemma 6.1](#) and is similar to the one used in [\[GLM⁺15\]](#); however, our analysis is quite different from theirs. Indeed, while they also obtain a similar decomposition theorem, the error guarantee is not exponentially small as we obtain and is needed in our application.

The formal description of the algorithm is at the end of this subsection. For exposition, we depict a labeled execution tree of the algorithm [Figure 1](#). `DECOMPOSE` is the main procedure that takes as input a rectangle $\mathcal{A} \times \mathcal{B} \subseteq [q]^n \times [q]^n$. This rectangle will always satisfy the invariant of having an *aligned* set of nodes *fixed* - i.e. there is an explicitly identified set of indices $F \subseteq [n]$ such that for some two fixed strings α, β , for all $(x, y) \in \mathcal{A} \times \mathcal{B}$, $x_F = \alpha$ and $y_F = \beta$. Observe that in the beginning, $\mathcal{A} = \mathcal{B} = [q]^n$ and $F = \emptyset$.

Each time `DECOMPOSE` is invoked by the algorithm, we create a new node in the execution tree and identify it as being created by `DECOMPOSE` by indexing it with v_1, v_2, \dots . If the set of fixed blocks F in the input rectangle \mathcal{R} has size $\geq d$, the algorithm terminates and adds the associated rectangle \mathcal{R} to Error_b ; Error_b contains the set of rectangles that account for error owing to the number of fixed blocks in them exceeding d . Next, if $\mu(\mathcal{R}) < \delta$, then \mathcal{R} is added to Error_a ; Error_a maintains the collection of rectangles that are labeled as error because their measure was too small.

Now, suppose that the input rectangle \mathcal{R} does not satisfy the conditions of Error_a or Error_b . If $\mu_{|\mathcal{R}}$ is `BLOCKWISE-DENSE`, then, we terminate the algorithm and return \mathcal{R} . Otherwise, there exists $S \subseteq [n]$ and some assignment to variables in S , say α_S such that $\mathbb{P}_{\mu_{|\mathcal{R}}}[X_S = \alpha_S] > q^{-0.8|S|}$ (or $\mathbb{P}_{\mu_{|\mathcal{R}}}[Y_S = \alpha_S] > q^{-0.8|S|}$). The idea is to split the rectangle $\mathcal{A} \times \mathcal{B}$ into two rectangles, $\mathcal{A}_{|S=\alpha_S} \times \mathcal{B}$ and $\mathcal{A}_{|S \neq \alpha_S} \times \mathcal{B}$; here, we define $\mathcal{A}_{|S=\alpha_S}$ denotes the set,

$$\mathcal{A}_{|S=\alpha_S} = \mathcal{A} \cap \{x : x_S = \alpha_S\} \text{ and } \mathcal{A}_{|S \neq \alpha_S} = \mathcal{A} \cap \{x : x_S \neq \alpha_S\}.$$

In the rectangle $\mathcal{A}_{|S=\alpha_S} \times \mathcal{B}$, X and Y don't have the same set of fixed blocks, since X is fixed in $F \cup S$ while Y is fixed only on F . To remedy this, the subroutine `XDECOMPOSE` (or `YDECOMPOSE`, respectively) is executed on the rectangle $\mathcal{A}_{|S=\alpha_S} \times \mathcal{B}$. The `DECOMPOSE` routine continues with the remaining rectangle $\mathcal{A}_{|S \neq \alpha_S} \times \mathcal{B}$. Each call to `XDECOMPOSE` or `YDECOMPOSE` is denoted by a node in the execution tree labeled by w_1, w_2, \dots .

The subroutine `XDECOMPOSE` (the case of `YDECOMPOSE` is analogous) takes as input the rectangle $\mathcal{A} \times \mathcal{B}$ along with the fixed set F that was the current input of the `DECOMPOSE` routine when `XDECOMPOSE` was invoked in addition to the new set of indices S that violated blockwise-density. `XDECOMPOSE` then chooses every possible value β for Y_S and for each β , calls `DECOMPOSE` recursively with the rectangle $\mathcal{A} \times \mathcal{B}_{|S=\beta}$ with $F \cup S$ as the set of fixed coordinates.

Decomposition Algorithm

Setup:

- A probability density μ such that $H_\infty(\mu|_I) \geq 1.9 \log q \cdot |I| - t$ for all $I \subseteq [n]$.
- A parameter $d \in \mathbb{N}$ and let $\delta \stackrel{\text{def}}{=} q^{-0.05d}$. Set $\text{Error}_a = \text{Error}_b = \emptyset$.

Decompose($\mathcal{A} \times \mathcal{B}, F$)

Input: A rectangle $\mathcal{A} \times \mathcal{B} \subseteq [q]^n \times [q]^n$, $F \subseteq [n]$: subset of “fixed” indices.

Invariant: $\mathcal{A}_F, \mathcal{B}_F$ are fixed.

1. If $|F| \geq d$, TERMINATE after setting $\text{Error}_d \leftarrow \text{Error}_d \cup \mathcal{A} \times \mathcal{B}$.
2. Set $\mathcal{R} \equiv \mathcal{R}_0 \equiv \mathcal{A} \times \mathcal{B}$.
3. While $\mu(\mathcal{R}) \geq \delta \cdot \mu(\mathcal{R}_0)$ do

(a) Let $(X, Y) \sim \mu|_{\mathcal{R}}$. If $X_{\bar{F}}, Y_{\bar{F}}$ are BLOCKWISE-DENSE, TERMINATE.

(b) Else, if there is an $S \subseteq [n] \setminus F$ and $\alpha \in [q]^S$ such that

$$\mathbb{P}[X_S = \alpha] > q^{-0.8|S|},$$

call YDECOMPOSE on input $(\mathcal{A}_{|S=\alpha}, \mathcal{B}, F, S)$ and set $\mathcal{R} \leftarrow \mathcal{A}_{|S \neq \alpha} \times \mathcal{B}$

(c) Else, if $\mathbb{P}[Y_S = \alpha] > q^{-0.8|S|}$ then call XDECOMPOSE on input $(\mathcal{A}, \mathcal{B}_{|S=\alpha}, F, S)$ and set $\mathcal{R} \leftarrow \mathcal{A} \times \mathcal{B}_{|S \neq \alpha}$.

4. Set $\text{Error}_a \leftarrow \text{Error}_a \cup \mathcal{R}$.

XDecompose($\mathcal{A} \times \mathcal{B}, F, S$)

Input: A rectangle $\mathcal{A} \times \mathcal{B}$, $F \subseteq [n]$: common subset of “fixed” indices; S : the set of coordinates newly fixed in \mathcal{A} (but not in \mathcal{B}).

Invariant: X_F and $Y_{F \cup S}$ are fixed.

1. For every $\beta \in [q]^S$, DECOMPOSE($\mathcal{A}_{|S=\beta}, \mathcal{B}, F \cup S$).

YDecompose($\mathcal{A} \times \mathcal{B}, F, S$)

Input: A rectangle $\mathcal{A} \times \mathcal{B}$, $F \subseteq [n]$: common subset of “fixed” indices.

S : the set of coordinates newly fixed in \mathcal{B} (but not in \mathcal{A}).

Invariant: $X_{F \cup S}$ and Y_F are fixed.

1. For every $\beta \in [q]^S$, DECOMPOSE($\mathcal{A}, \mathcal{B}_{|S=\beta}, F \cup S$).

6.4 Analysis: Proof of Theorem 6.4

We now analyse the algorithm.

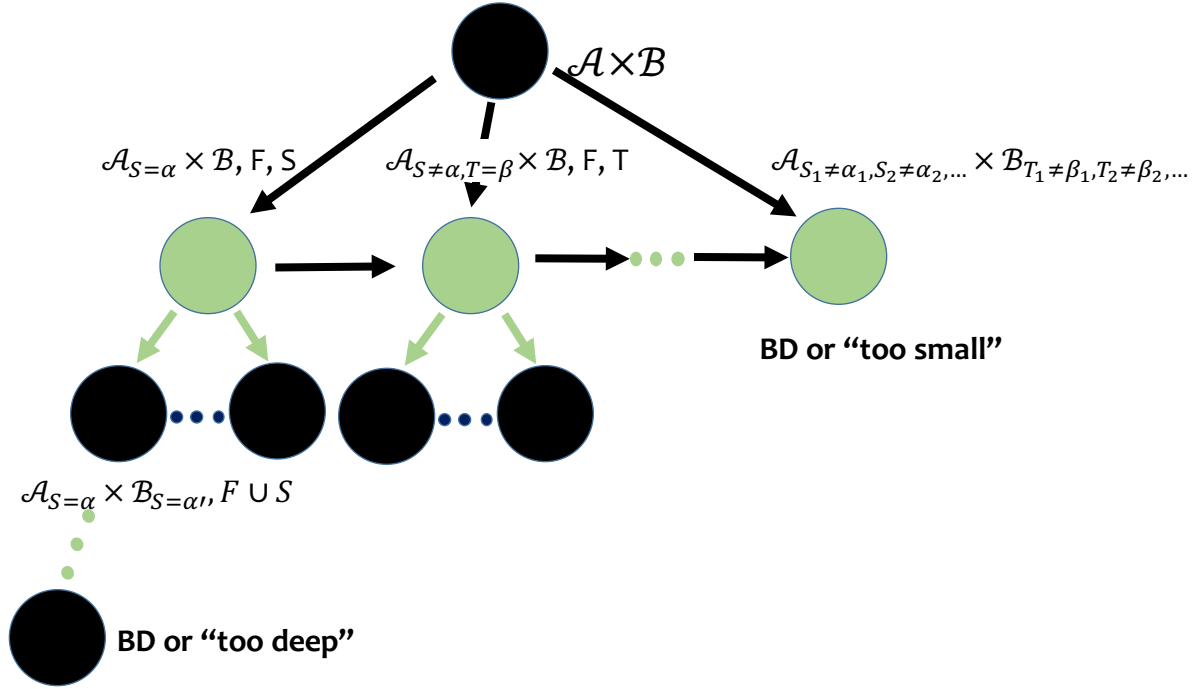


Figure 1: Execution Tree: Blue Nodes are Calls of `DECOMPOSE`, Red Nodes are calls of `XDECOMPOSE` or `YDECOMPOSE`. “too deep” $\equiv \text{Error}_a$, “too small” $\equiv \text{Error}_b$ and BD = BLOCKWISE-DENSE

Execution Tree. Consider the execution tree of the decomposition algorithm (Figure 1). We associate a node ρ corresponding to a call of `DECOMPOSE` with parameters $\mathcal{A}_\rho, \mathcal{B}_\rho$ and F_ρ for $\mathcal{A}_\rho, \mathcal{B}_\rho \subseteq [q]^n$ and $F_\rho \subseteq [n]$. Nodes ρ corresponding to a call of `XDECOMPOSE` or `YDECOMPOSE` are associated with an additional parameter $S_\rho \subseteq [n]$. The calls of `DECOMPOSE` and `XDECOMPOSE` or `YDECOMPOSE` alternate.

Before we analyze the decomposition, we introduce some notation.

- For each vertex ρ , let $\mu(\rho) = \mu(\mathcal{A}_\rho \times \mathcal{B}_\rho)$. Let $R(\rho)$ denote the rectangle $\mathcal{A}_\rho \times \mathcal{B}_\rho$.
- For a child b of a vertex a , let $\mu(b|a) = \mu(b)/\mu(a)$.
- We will reserve the letter v (and various suffixes) for nodes corresponding to calls of `DECOMPOSE` (i.e., the vertices in odd-layers) and the letter w for nodes corresponding to calls of `XDECOMPOSE` or `YDECOMPOSE` (i.e., the vertices in even-layers).
- For a vertex v let $C_v = (w_1, \dots, w_{c_v})$ be the children of v (in the order they were generated), where we assume that w_1, \dots, w_{c_v-1} lead to recursive calls of `XDECOMPOSE` or `YDECOMPOSE` while w_{c_v} corresponds to the rectangle marked as Error_a in Step(4) of `DECOMPOSE`.
- For a vertex v , and $w_i \in C(v)$, let $S_{w_i} \subseteq [n]$ denote the corresponding set of new blocks that were fixed to produce w_i .

- We say a node v in the tree is *bad* if $|F_v| \geq 2d$ but $\mu_{|\mathcal{A}_v \times \mathcal{B}_v}$ is not BLOCKWISE-DENSE in the non-fixed blocks. A root-to-leaf path $v_1, w_1, v_2, w_2, \dots, v_t, w_t, v_{t+1}$ is called a *bad path* if v_{t+1} is bad.

Note that the tree yields a partition of $[q]^n \times [q]^n$ as,

$$[q]^n \times [q]^n = \left(\bigcup_{v \in \mathcal{L}} \mathcal{R}(v) \right) \cup \text{Error}_a \cup \text{Error}_b,$$

where \mathcal{L} denotes the leaves of the tree not added to Error_a or Error_b . We will show that this partition satisfies the conditions of the theorem when we take $\text{Error} = \text{Error}_a \cup \text{Error}_b$. Note that, all the leaves in the execution tree correspond to the calls of DECOMPOSE.

We first argue that $\mu_{|\mathcal{R}(v)}$ for $v \in \mathcal{L}$ gives d -CBD densities in the non-fixed coordinates.

Lemma 6.5 (Good Rectangles). *Let $v \in \mathcal{L}$. Then, $\mu_{|\mathcal{R}(v)}$ is an aligned d -CBD.*

Proof. The assumptions imply that $|F_v| \leq d$. Further, as $\mathcal{R}(v)$ was not added to Error_a , $\mathcal{R}(v)$ is a leaf because for $(X, Y) \sim \mu_{|\mathcal{R}(v)}$, $X_{\bar{F}}, Y_{\bar{F}}$ are BLOCKWISE-DENSE. Thus, $\mu_{|\mathcal{R}(v)}$ is an aligned d -CBD density. \square

It is also easy to bound $\mu(\text{Error}_a)$:

Lemma 6.6. $\mu(\text{Error}_a) \leq d\delta$.

Proof. Note that each call of DECOMPOSE leads to at most one rectangle being added to Error_a . Further, if $\text{DECOMPOSE}(R_0)$ led to adding a rectangle $\mathcal{R} \subseteq R_0$ to Error_a , then $\mu(\mathcal{R}) < \delta \cdot \mu(R_0)$. Now, in any single layer of the execution tree the nodes corresponding to calls of DECOMPOSE are associated with disjoint rectangles. Thus, the total measure of all the rectangles that are included in Error_a due to DECOMPOSE calls from nodes in a specific layer is at most δ . As there are at most d layers that have DECOMPOSE nodes, it follows that $\mu(\text{Error}_a) \leq d\delta$. \square

The main task is to prove an upper bound on $\mu(\text{Error}_b)$.

Bounding $\mu(\text{Error}_b)$. We begin with an important definition.

Definition 6.7 ($\theta(w_i | v)$). For a node v associated with a DECOMPOSE call, let $C_v = (w_1, \dots, w_{c_v})$ be the children of v . Define

$$\theta(w_i | v) = \sum_{j=i}^{c_v} \mu(w_j | v).$$

Intuitively, $\theta(w_i | v)$ denotes the relative measure of the rectangle \mathcal{R} inside Step 4 of Algorithm 6.3 just before the call of XDECOMPOSE or YDECOMPOSE associated with the node w_i .

Lemma 6.8. *Fix any vertex v and a child w of v ,*

$$\mu(w|v) \geq q^{-0.8|S_w|} \cdot \theta(w|v).$$

Proof. Let \mathcal{R} be the rectangle processed in the while loop inside the call of `DECOMPOSE` associated with v just before the call to create w . Suppose that w was created by a recursive call to `XDECOMPOSE` for $S_w \subseteq [n]$ such that $\mathbb{P}_{(X,Y) \sim \mu_{|\mathcal{R}|}}[X_{S_w} = \alpha] \geq q^{-0.8|S_w|}$ (the case of a call to `YDECOMPOSE` can be dealt with analogously). Thus, $\mu_{|\mathcal{R}|}(\mathcal{R}(w)) \geq q^{-0.8|S_w|}$.

Now, we can write $\mu(w|v)$ as a product of the relative probability of \mathcal{R} in $\mathcal{R}(v)$ and the relative probability of $\mathcal{R}(w)$ under \mathcal{R} . Concretely,

$$\mu(w|v) = \mu_{|\mathcal{R}(v)|}(\mathcal{R}(w)) = \mu_{|\mathcal{R}(v)|}(\mathcal{R}) \cdot \mu_{|\mathcal{R}|}(\mathcal{R}(w)) = \theta(w|v) \cdot \mu_{|\mathcal{R}|}(\mathcal{R}(w)) \geq \theta(w|v) \cdot q^{-0.8|S_w|},$$

where the last equality follows from the definition of $\theta(w|v)$. □

Next, we estimate $\mu(v)$ for every v that is a *bad* leaf i.e., a leaf at depth $2d + 1$. We begin by showing a bound on $\mu(v)$ for an arbitrary node v .

Lemma 6.9. *For any vertex v in the execution tree,*

$$\mu(v) \leq 2^t \cdot q^{-1.9|F_v|}.$$

Proof. The rectangle $\mathcal{R}(v)$ corresponding to v has the blocks F_v fixed, while the values in the remaining blocks could also be constrained. Let us suppose $X_{F_v} = \alpha$ and $Y_{F_v} = \beta$ for $(X, Y) \in \mathcal{R}(v)$. By our assumption on μ ,

$$H_\infty(\mu_{|F_v|}) \geq 1.9 \cdot \log q \cdot |F_v| - t,$$

which implies that,

$$\mu(v) \leq \mathbb{P}_\mu[X_{F_v} = \alpha \wedge Y_{F_v} = \beta] \leq 2^{-H_\infty(\mu_{|F_v|})} \leq 2^t \cdot q^{-1.9|F_v|}.$$

□

Lemma 6.10. *Let $v_1, w_1, \dots, v_\ell, w_\ell, v_{\ell+1}$ be a bad path ending at a node that is added to `Errorb`. Then,*

$$\mu(v_{\ell+1}) \leq q^{-1.1|F_{v_{\ell+1}}|} \cdot 2^t \cdot \prod_{i=1}^{\ell} \frac{\mu(w_i|v_i)}{\theta(w_i|v_i)}$$

Proof. Let $s = |F_{v_{\ell+1}}|$; then, by [Lemma 6.9](#),

$$\mu(v_{\ell+1}) \leq q^{-1.9s} \cdot 2^t$$

On the other hand, we know that $\sum_{i=1}^{\ell} |S_{w_i}| = s$. Thus, by [Lemma 6.8](#),

$$\mu(v_{\ell+1}) = \prod_{i=1}^{\ell} \mu(v_{i+1}|w_i) \mu(w_i|v_i) \geq \prod_{i=1}^{\ell} \mu(v_{i+1}|w_i) \cdot \theta(w_i|v_i) \cdot q^{-0.8|S_{w_i}|}.$$

The above two inequalities imply that,

$$\prod_{i=1}^{\ell} \mu(v_{i+1}|w_i) \theta(w_i|v_i) \leq q^{-1.1s} \cdot 2^t.$$

The claim now follows from the above inequality along with

$$\mu(v_{\ell+1}) = \prod_{i=1}^{\ell} \mu(v_{i+1}|w_i) \mu(w_i|v_i) = \left(\prod_{i=1}^{\ell} \mu(v_{i+1}|w_i) \theta(w_i|v_i) \right) \cdot \prod_{i=1}^{\ell} \frac{\mu(w_i|v_i)}{\theta(w_i|v_i)}.$$

□

We need the following elementary lemma.

Lemma 6.11. *Let $a_1, \dots, a_N \in (0, 1)$ be such that $\sum_{i=1}^N a_i = 1$ and $a_{N-1} + a_N \geq \varepsilon$. Then,*

$$\sum_{j=1}^N \frac{a_j}{\sum_{i \geq j} a_i} \leq \lceil \log(1/\varepsilon) \rceil + 2.$$

Proof. For $i < N$, let $s_i = \sum_{j \geq i} a_j$; clearly, s_i is a decreasing sequence and $s_{N-1} > \varepsilon$. Let ℓ_i be the largest index such that $s_{\ell_i} = \sum_{j=\ell_i}^N a_j \geq \frac{1}{2^i}$.

By definition,

$$\sum_{j=\ell_i+1}^{\ell_{i+1}} a_j \leq \sum_{j=\ell_i+1}^N a_j < \frac{1}{2^i}$$

Let $t = \lceil \log(1/\varepsilon) \rceil$. Clearly, $\ell_t \geq N - 1$. Now, we have,

$$\begin{aligned} \sum_{i=1}^{N-1} \frac{a_i}{s_i} &= \sum_{i=0}^t \sum_{j=\ell_i+1}^{\ell_{i+1}} \frac{a_j}{s_j} \\ &\leq \sum_{i=0}^t \frac{\sum_{j=\ell_i+1}^{\ell_{i+1}} a_j}{s_{\ell_{i+1}}} \\ &\leq \sum_{i=0}^t 1 = t + 1. \end{aligned}$$

□

We have the following immediate consequence of the above lemma.

Corollary 6.12. *For every vertex v ,*

$$\sum_{w \in C_v} \frac{\mu(w|v)}{\theta(w|v)} \leq \lceil \log(1/\delta) \rceil + 2.$$

Proof. Let $C_v = (w_1, \dots, w_{c_v})$. Then, $\theta(w_{c_v-1}|v) = \mu(w_{c_v-1}|v) + \mu(w_{c_v}|v) \geq \delta$. As $\theta(w_i|v) = \sum_{j \geq i} \mu(w_j|v)$, the claim now follows by applying the previous lemma to the numbers $\mu(w_1|v), \mu(w_2|v), \dots, \mu(w_{c_v}|v)$.

□

Lemma 6.13. *The sum over all leaves*

$$\sum_{\text{paths } v_1, w_1, \dots, w_{\ell-1}, v_{\ell}} q^{-|F_{v_{\ell}}|} \prod_{i=1}^{\ell} \frac{\mu(w_i|v_i)}{\theta(w_i|v_i)} \leq (\lceil \log 1/\delta \rceil + 2)^d$$

Proof. For each v corresponding to a `DECOMPOSE` call, define a probability distribution $\gamma(\cdot|v)$ over C_v as follows: for each $w \in C_v$,

$$\gamma(w|v) = \frac{\mu(w|v)}{\theta(w|v)} \cdot \left(\sum_{w \in C_v} \frac{\mu(w|v)}{\theta(w|v)} \right)^{-1}.$$

For each w corresponding to a `XDECOMPOSE` or `YDECOMPOSE` call consider the uniform distribution over the children of w . Then, these distributions induce a probability distribution over the leaves of the execution tree: the probability of any leaf $v_{\ell+1}$ is given by $\prod_{i=1}^{\ell} q^{-|S_{w_i}|} \cdot \gamma(w_i|v_i)$, where $v_1, w_1, \dots, v_{\ell}, w_{\ell}, v_{\ell+1}$ is the path from the root to $v_{\ell+1}$.

By our construction, the total probability under the above distribution of all the leaves is 1. Thus, the sum over all leaves

$$\sum_{\text{paths } v_1, w_1, \dots, v_{\ell}, w_{\ell}} \prod_{i=1}^{\ell} q^{-|S_{w_i}|} \cdot \gamma(w_i|v_i) = 1.$$

By Corollary 6.12, $\gamma(w_i|v_i) \geq \frac{\mu(w_i|v_i)}{\theta(w_i|v_i)} \cdot \frac{1}{(\lceil \log 1/\delta \rceil + 2)}$. The result follows from substituting this into the previous expression and using the fact that $\ell \leq d$ (number of fixed coordinates is at most d). \square

We are now ready to bound $\mu(\text{Error}_b)$.

Lemma 6.14. $\mu(\text{Error}_b) \leq q^{-0.1d} \cdot 2^t \cdot (\lceil \log 1/\delta \rceil + 2)^d$.

Proof. Let Bad denote all leaves that resulted in rectangles being added to Error_b . The proof follows by using Lemmas 6.10 and Lemma 6.13. For brevity, in the following, let $v_1, w_1, \dots, v_{\ell}, w_{\ell}, v_{\ell+1} = v$ be the path to v from the root.

$$\begin{aligned} \sum_{v \in Bad} \mu(v) &\leq \sum_{v \in Bad} q^{-1.1|F_v|} \cdot 2^t \cdot \prod_{i=1}^{\ell} \frac{\mu(w_i|v_i)}{\theta(w_i|v_i)} \\ &\leq q^{-0.1d} \cdot 2^t \left(\sum_{v \in Bad} q^{-|F_v|} \cdot \prod_{i=1}^{\ell} \frac{\mu(w_i|v_i)}{\theta(w_i|v_i)} \right) \leq q^{-0.1d} \cdot 2^t \cdot (\lceil \log 1/\delta \rceil + 2)^d. \end{aligned}$$

\square

Proof of Theorem 6.4. One executing Algorithm 6.3, we obtain a partition of $[q]^n \times [q]^n$ into $\cup_{v \in \mathcal{L}} \mathcal{R}(v) \cup \text{Error}$, where we define $\text{Error} = \text{Error}_a \cup \text{Error}_b$.

By Lemma 6.5, $\mu_{|\mathcal{R}(v)}$ is an aligned d -CBD for each $v \in \mathcal{L}$. Furthermore, the total measure of Error is

$$\begin{aligned} \mu(\text{Error}) &= \mu(\text{Error}_a) + \mu(\text{Error}_b) \\ &\leq d\delta + q^{-0.1d} \cdot 2^t \cdot (\lceil \log 1/\delta \rceil + 2)^d, \quad (\text{Lemma 6.6 and Lemma 6.14}) \\ &\leq 2^t \cdot (dq^{-0.05})^d \quad (\text{for } \delta = q^{-0.05d}). \end{aligned}$$

This completes the proof. \square

7 Non-negative Rank and LP Lower Bounds for CSPs

In this section, we show that proving a lower bound on the size of linear programming relaxations for CSPs reduces to proving non-negative rank lower bound on a pattern matrix M_f for an appropriate choice of f . We then use this characterization to prove [Theorem 1.2](#), [Corollary 1.3](#). We first show the following.

Lemma 7.1 (LP Lower Bounds from Pattern Matrices). *Let $0 < s < c \leq 1$ and let $\Lambda : \{-1, 1\}^k \rightarrow \{0, 1\}$ be a predicate and let I^* be an instance of $\text{CSP}(\Lambda)$ on n variables such that $\text{opt}(I^*) \leq s$ and let $f(x) = c - I^*(x)$. For all $g : [q] \times [q] \rightarrow \{-1, 1\}$, any linear programming relaxation of $\text{CSP}(P)$ that achieves (c, s) -approximation on instances with $q \cdot n$ variables, has size at least $R \geq \Omega(\text{rank}_+(M_f^g))$.*

The above lemma is an easy consequence of the characterization of size of linear programs for CSPs [[CLRS13](#), [Yan88](#)]. Towards stating the characterization, let us fix a CSP Λ . For $n \in \mathbb{N}$ and $s \in [0, 1]$, let Λ_n^s denote the family of all instances of the CSP on n variables with $\text{opt}(I) \leq s$. With these definitions, we are ready to state the characterization.

Fact 7.2. [[CLRS13](#)] *The size of the smallest linear program that (c, s) -approximates a CSP Λ on instances with n variables is $\Theta(\text{rank}_+(\mathcal{M}_{n,s}))$ where the matrix $\mathcal{M}_{n,s} : \Lambda_n^s \times \{-1, 1\}^n \rightarrow \mathbb{R}$ is defined as*

$$\mathcal{M}_{n,s}(I, x) \stackrel{\text{def}}{=} c - I(x).$$

To prove [Lemma 7.1](#), we will show that M_f^g is a sub-matrix of $\mathcal{M}_{q \cdot n, s}$, and therefore $\text{rank}_+(\mathcal{M}_{q \cdot n, s}) \geq \text{rank}_+(M_f)$. By [Fact 7.2](#), this implies that the size of any LP that (c, s) -approximates the CSP on instances with $q \cdot n$ is at least $\text{rank}_+(M_f)$. Hence, [Lemma 7.1](#) is an immediate consequence of the following.

Lemma 7.3. *Let I^* be an instance of CSP on n variables such that $\text{opt}(I^*) = s$ and let $f(x) = c - I^*(x)$. Let $g : [q] \times [q] \rightarrow \{-1, 1\}$ be a gadget function. Then, M_f^g is a sub-matrix of $\mathcal{M}_{q \cdot n, s}$.*

Proof. For $\alpha \in [q]$, define its g -encoding $g \circ \alpha \in \{-1, 1\}^q$ consisting of the truth-table of the function $g \circ \alpha : \beta \rightarrow g(\alpha, \beta)$. Specifically, if we index the coordinates of $g \circ \alpha$ with $\beta \in [q]$, then $(g \circ \alpha)_\beta \stackrel{\text{def}}{=} g(\alpha, \beta)$. Similarly for $x \in [q]^n$, define $\tilde{x} \in \{-1, 1\}^{q \cdot n}$ by setting $\tilde{x}_i := g \circ x_i$.

For each $y \in [q]^n$, we create an instance \mathcal{I}_y on $N = q \cdot n$ variables. We will index the variables of \mathcal{I}_y by $[n] \times [q]$ and denote them by $\{z_{i,\beta} | i \in [n], \beta \in [q]\}$. The instance \mathcal{I}_y is obtained by planting the instance I^* on the subset of variables $\{z_{1,y_1}, \dots, z_{1,y_n}\}$.

By definition of the matrices \mathcal{M} and M_f^g , we conclude that for any $x, y \in [q]^n$,

$$\begin{aligned} \mathcal{M}_{q \cdot n, s}(\mathcal{I}_y, \tilde{x}) &= c - \mathcal{I}_y(\tilde{x}), \\ &= c - I^*(\tilde{x}_{1,y_1}, \dots, \tilde{x}_{1,y_n}), \\ &= c - I^*((g \circ x_1)_{y_1}, \dots, (g \circ x_n)_{y_n}), \\ &= c - I^*(g(x_1, y_1), \dots, g(x_n, y_n)) = f(g^n(x, y)), \\ &= M_f^g(x, y). \end{aligned}$$

Therefore, the matrix M_f^g is a sub-matrix of $\mathcal{M}_{q \cdot n, s}$ as desired. \square

Now we are ready to wrap up the proof of our main result [Theorem 1.2](#), concerning the optimality of Sherali-Adams linear programs, among all linear programs of roughly the same size.

Proof of Theorem 1.2. Suppose $f(n)$ -round Sherali-Adams relaxation for a CSP Λ does not achieve a (c, s) -approximation on instances with n variables. This implies that there exists instances \mathcal{I}_n on n variables such that $\text{opt}(\mathcal{I}) \leq s$ but the optimum value of $f(n)$ -round Sherali-Adams linear program $\text{opt}_{\text{SA}(f(n))}(\mathcal{I}) > c$.

Set $I_n(x) := c - \frac{1}{n} - \mathcal{I}_n(x)$. The work of Chan et al. [[CLRS13](#)] observes that the dual to the $f(n)$ -round Sherali-Adams linear program corresponds to expressing the function $c - \mathcal{I}_n(x)$ as a sum of non-negative $f(n)$ -juntas. In particular, this implies that $\deg_+(I_n + \frac{1}{n}) \geq f(n)$.

Applying [Theorem 1.7](#) to function h_n we get that,

$$\text{rank}_+(M_{\mathcal{I}_n}^b) \geq 2^{\Omega(b \cdot \deg_+(I_n + \frac{1}{n}))}$$

for some $b = \Theta(\log n)$. By [Lemma 7.3](#), the matrix $M_{\mathcal{I}_n}^b$ is a sub-matrix $\mathcal{M}_{n,q,s}$ for $q = 2^b$. Therefore we get,

$$\text{rank}_+(\mathcal{M}_{n^H,s}) \geq n^{h \cdot \deg_+(I_n + \frac{1}{n})} \geq n^{h \cdot f(n)},$$

for some constants $h, H \in \mathbb{N}$. Using [Lemma 7.1](#), this implies that no linear program of size $n^{h \cdot f(n)}$ can $(c - \frac{1}{n}, s)$ -approximate the CSP Λ on instances with n^H variables. \square

We now prove [Theorem 1.2](#), [Corollary 1.3](#). For this, we need the following results on the performance of the Sherali-Adams hierarchy for CSPs. Charikar et. al. [[CMM09](#)] showed the following lower bound for MAX-CUT.

Theorem 7.4 (Sherali Adams Integrality Gaps [[CMM09](#)]). *For every $\varepsilon > 0$, there is a $\gamma = \gamma(\varepsilon)$ such that the n^γ -round Sherali-Adams relaxation for MAX-CUT does not achieve a $(1/2 + \varepsilon, 1 - \varepsilon)$ -approximation⁷.*

Grigoriev [[Gri01](#)] showed a lower bound for the *Sum-of-Squares SDP hierarchy* (that is a strengthening of the Sherali-Adams LP hierarchy and thus the lower bounds carry over) for 3XOR; Schoenebeck [[Sch08a](#)] rediscovered this result and also observed that it implies a similar lower bound for the MAX-3SAT problem. Following this, [[BGMT12](#)] extended this result to show a $\Omega(n)$ -round lower bound for every *pairwise independent* CSP; here, a CSP defined by a predicate $P : \{0, 1\}^k \rightarrow \{0, 1\}$ is pairwise independent if there exists a balanced pairwise independent distribution μ supported on $P^{-1}(1)$.

Theorem 7.5 ([[Gri01](#), [Sch08a](#), [BGMT12](#)]). *For every k -ary pairwise independent predicate P and $\varepsilon > 0$, there exists a constant $c = c(k, \varepsilon)$ such that the cn -round Sherali-Adams relaxation for MAX-CSP problem on predicate P achieves a $(|P^{-1}(1)|/2^k + \varepsilon, 1 - \varepsilon)$ -approximation. As a corollary, for some constants $c_1(\varepsilon), c_2(\varepsilon)$, the $c_1(\varepsilon)$ -round Sherali-Adams relaxation for MAX-3SAT does not achieve a $(7/8 + \varepsilon, 1 - \varepsilon)$ -approximation, and $c_2(\varepsilon)$ -round Sherali-Adams relaxation for MAX-3XOR does not achieve a $(1/2 + \varepsilon, 1 - \varepsilon)$ -approximation.*

⁷The results of [[CMM09](#)] are actually stated in terms of integrality gaps, but their proofs actually show this stronger statement.

Proof of Corollary 1.3. Consider the case of MAX-3SAT. Then, combining the above theorem with Theorem 1.2 we get that any LP relaxation for MAX-3SAT of size $n^{hc_1(\varepsilon)n}$ on n^H -variables does not achieve a $(7/8 + \varepsilon, 1 - \varepsilon)$ -approximation. Let $N = n^H$. Then, this says that no LP relaxation for MAX-3SAT of size $N^{hc_1(\varepsilon)N^{1/H}/H} = N^{\Omega_\varepsilon(N^{1/H})}$ achieves a $(7/8 + \varepsilon, 1 - \varepsilon)$ -approximation. The latter condition in particular implies that such LP relaxations have integrality gap at least $1 - \varepsilon/(7/8 + \varepsilon) = 8/7 - O(\varepsilon)$. This implies the claimed lower bound for MAX-3SAT. The claims for MAX-3XOR, MAX-CUT, and pairwise independent predicates follow similarly. \square

Proof of Theorem 1.8. We only sketch the argument here and refer to [CLRS13] where the connection between such separations and lower bounds for CSPs as above is drawn out in more detail.

The theorem essentially follows by showing corresponding separations for *degrees* and using our lifting theorem. Let $f : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ be a function and let $\deg(f)$ be the degree of f as a polynomial and for all $\delta > 0$, let

$$\deg_+^\delta(f) = \min\{\deg_+(h) : \|h - f\|_\infty \leq \delta\|f\|_\infty\}.$$

Then, for all b , $\text{rank}(M_f^b) \leq \binom{n}{\deg(f)} \cdot 2^{b \cdot \deg(f)}$, $\text{rank}_+^\delta(M_f^b) \leq \binom{n}{\deg_+^\delta(f)} \cdot 2^{b \cdot \deg_+^\delta(f)}$. Thus, to show the theorem, it would suffice to find a function $f : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathbb{E}[f] = \Omega(1)$, $\deg(f) = O(1)$, $\deg_+^\varepsilon(f) = O(\log(1/\varepsilon))$ and $\deg_+(f + \varepsilon) = \Omega_\varepsilon(n)$. For, if we take $M = M_f^b$ for $b \geq C \log n$ as in Theorem 1.7, then $M \in \mathbb{R}_{\geq 0}^{N \times N}$ for $N = 2^{bn}$ and

$$\begin{aligned} \text{rank}(M) &\leq \binom{n}{\deg(f)} \cdot 2^{b \cdot \deg(f)} = (\log N)^{O(1)} \\ \text{rank}_+^\varepsilon(M) &\leq \binom{n}{\deg_+^\varepsilon(f)} \cdot 2^{b \cdot \deg_+^\varepsilon(f)} = n^{O(\log(1/\varepsilon))} = (\log N)^{O(\log(1/\varepsilon))}, \\ \text{rank}_+(M) &= \exp(\Omega(b \cdot \deg_+(f + O(1/n)))) = \exp(\Omega(b \cdot \deg_+(f + \varepsilon))) = N^{\Omega_\varepsilon(1)}. \end{aligned}$$

To show the existence of such a function, let \mathcal{I} be an instance of MAX-3SAT on n -variables such that $\text{opt}(\mathcal{I}) \leq 7/8 + \varepsilon$, but the $\Omega_\varepsilon(n)$ -round Sherali-Adams relaxation has value at least $1 - \varepsilon$. Such instances exist by [Gri01, Sch08a]. Define $f : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$ as $f(x) = 1 - 2\varepsilon - \mathcal{I}(x)$. (This is non-negative valued for $\varepsilon < 1/24$.)

Clearly, $\deg(f) = 3$ and by the relation between Sherali-Adams relaxations and \deg_+ , $\deg_+(f + \varepsilon) = \deg_+(1 - \varepsilon - \mathcal{I}(\cdot)) = \Omega_\varepsilon(n)$. To finish the proof, it remains to show that $\deg_+^\varepsilon(f) = O(\log(1/\varepsilon))$. This follows from a similar argument used in [CLRS13] for MAX-CUT. \square

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